# GRAPHICAL DETERMINATION

OF

# EARTH SLOPES, RETAINING WALLS AND DAMS

BY

CHARLES PRELINI, C.E.

PROFESSOR OF CIVIL ENGINEERING, MANHATTAN COLLEGE, NEW YORK CITY

Author of *"Earth and Rock Excavation," "Tunneling,"* etc.

NEW YORK

D. VAN NOSTRAND COMPANY

23 MURRAY AND 1908 27 WARREN STS

*The Plimpton Press Norwood Mass. U.S.A.*

# PREFACE

A large part of this work consists of graphical methods of solving problems concerning the slopes of earth embankments, the lateral pressure of earth against a wall, and the thickness of retaining walls and dams. The graphical methods of Culmann, Rebhann, Weyrauch, Blanc, and others have been employed ; the general course of the discussion is similar to that followed by Professor Senesi, of Italy. Hence, with the exception of the graphical determination of earth slopes of uniform stability, there is nothing in these pages which has not been already published and criticised. But the prominence given to the graphical over the analytical mode of treatment may be found useful to a numerous class of students.

The book is divided into five chapters. In the first chapter attention is given to the forces which determine the various slopes of the earth embankments. Young engineers will find it profitable to study with care the economies secured by designing the slopes of deep trenches for equal stability instead of using the ordinary slope of 1 to 1.

The second chapter is devoted to the graphical determination of the pressure of earth against a retaining wall, following the theory of Professor Rebhann. This chapter also contains solutions of the different problems which may be encountered in practical work.

In the third chapter is given the analytical demonstration of Professor Rebhann's theory, together with the formulas

deduced from the analytical theories of Weyrauch and Rankine. In three tables a comparison is made between the results obtained by the graphical process of determining the earth pressure and those given by the analytical method. As will be seen, the results agree well, showing that for practical purposes both methods lead to almost the same result. The tables were deduced from the class work of the author's students at Manhattan College.

The fourth chapter is devoted to the design of retaining walls. Here will be found the most common types of wall used in practice, together with the manner of determining the thickness of their bases both graphically and analytically.

The fifth chapter is devoted to dams. The space devoted to a subject so popular and extensively discussed in schools and text-books may seem disproportionately small; but in the class room, dams should be taken for what they are; viz. a particular case of retaining walls in which the material to be sustained is deprived of friction.

The discussion of the reliability of the various theories is omitted in order to avoid confusion in the untrained mind of the student, who is unable to follow with profit such complicated discussions based upon slight differences in the assumptions. It is only when students have mastered the subject that they are able to discuss intelligently the various theories and to accept or discard them according to their comparative value. Such critical work should be done by the students individually and not collectively. To show young men at the very beginning of their professional studies that all theories are more or less defective would generate confusion, produce uncertainty and a want of self-confidence. It would lead to a depreciation of every theory and to undue reliance on practical formulas.

In conclusion, these pages are intended for students and not for professional engineers. Simplicity and clearness have been the main objects in view; the experience of the class room makes the author believe that this little work will be of use to students and teachers, while at the same time it may be of some help to the practical engineer.

C. PRELINI.

MANHATTAN COLLEGE, NEW YORK,
September, 1908.

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# EARTH SLOPES, RETAINING WALLS, AND DAMS

## CHAPTER I

### THE STABILITY OF EARTH SLOPES

1. **Measurement of Slopes.** — When a trench is to be cut in soil for temporary use, the sides of the excavation are kept vertical by sheeting or bracing. But when the trench is to remain as a permanent piece of work, as in the construction of roads, railroads, canals, etc., artificial means for holding up the sides are usually out of the question, and to prevent the fall of the sides of the trench they are cut with a slope sufficient to enable the earth to maintain itself in place without support.

The slope of a cut or embankment is represented by the ratio of the base to the height of the right triangle formed by the slope and a pair of horizontal and vertical lines. Thus, a slope of 1 to 1 indicates that the base of the slope is equal to the height; the slope of $1\frac{1}{2}$ to 1 that the base is one and a half times the height; 2 to 1 that the base is double the height; and so on. In Fig. 1, $AB$ represents the base and $BC$ the height of a cut; its slope would be indicated by the fraction $\frac{AB}{BC}$, or, if $BC = 1$, simply by $AB : 1$.

It is not possible to give a general rule for the slope to which trenches should be cut, since in each individual case

there are so many particular conditions to be taken into consideration. In important works the slopes of the trenches should be fixed, after careful examination with regard to all the local conditions. Only in minor works may simple practical rules be followed. Such are: the slope of $1\frac{1}{2}$ to 1 is usually adopted for trenches cut through loose soils; 1 to 1 for earth of ordinary consistency; 1 to 4 for soft rock; and 1 to 10 for hard and compact rock.

FIG. 1.

**2. Equilibrium of a Slope.** — In general it may be said that the slope of any earth embankment is the result of three different forces: (1) the weight of the material of the slope; (2) the friction between its particles; and (3) the cohesive forces in the material, which keep together its various particles.

**3. Weight of Soils.** — In the excavation of trenches through loose soils, it is the weight of the material that tends to cause the collapse of the sides. To determine the proper slope to be used, therefore, it is essential to know the weight of the material concerned.

The weight per cubic foot, or specific weight of earth or of any other substance, is obtained by weighing a mass of known dimensions and dividing by its volume. But earth when removed from its natural bed increases in volume, so that to correctly determine its specific weight it is necessary to measure very accurately the dimensions of the specimen mass of earth before excavation. Further, since moisture increases the weight of earth, the degree of its moisture content at the time the weight is determined should also be observed and recorded.

The following table gives the weight of some soils under different conditions of moisture content:

| Quality of Soil | | Weight of 1 Cu. Ft. in Lbs. |
|---|---|---|
| Common Loam | dry | 72 to 80 |
| | moist | 70 " 76 |
| | very moist | 70 " 76 |
| | full of water | 104 " 112 |
| Sand | dry | 90 " 106 |
| | very moist | 118 " 129 |
| Gravel | | 90 " 106 |

**4. Internal Friction of Soils.** — When a mass of soil is thrown into a pile, it will dispose itself with a certain maximum slope of surface, called the natural slope. This slope is characteristic for any particular material in a given condition, but varies with the nature of the soil and the amount of moisture which it contains.

In Fig. 2, let $D$ be one of the particles of the earth on the surface of the slope after the pile has assumed its natural slope at an angle $\phi$ with the horizontal. Let $W$ be the weight of the particle $D$. Resolve the weight into its two components $N$ and $F$ respectively normal to and parallel to the natural slope $AB$. The force $N$ presses the particle $D$ against the rest of the soil, while the force $F$ will have a tendency to push it down the slope. The action of the force $F$ is resisted by the friction between particle $D$ and the rest of the soil, under the pressure $N$. Since friction does not depend upon the magnitude of the surface of contact, but only upon

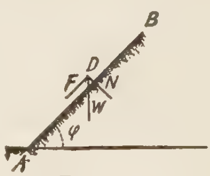


Fig. 2.

the total pressure, its relation to this pressure being expressed by a constant coefficient of friction $f$, we may conclude that the following relation is true:

$$F = fN. \tag{1}$$

But from the resolution of forces we know that

$$F = W \sin \phi$$

and

$$N = W \cos \phi.$$

Substituting these values in equation (1) gives

$$W \sin \phi = f W \cos \phi ;$$

or, in other terms,

$$f = \frac{\sin \phi}{\cos \phi} = \text{tang } \phi,$$

or the coefficient of friction is equal to the trigonometric tangent of the angle of natural slope of the earth.

The angle $\phi$ is called the angle of friction and is used to indicate the natural slope of the soils.

In practice the coefficient of friction of the soils is obtained by piling them up so that they may dispose themselves according to their natural slope without cohesive action, and observing the angle of this slope, given by the ration of the horizontal distance between any two points taken along the line of the natural slope, to their difference of level. This ratio is $\frac{1}{f}$ or $\cot \phi$.

In the following table are given the angles of natural repose, the coefficients of friction, and the slopes of various soils.

| Quality of Soil | | Angle of Repose | Coefficient of Friction | Slope of Repose |
|---|---|---|---|---|
| Sand | dry | 35° | 0.7002 | 1.4281 |
| | moist | 40° | 0.8391 | 1.1918 |
| | very wet | 30° | 0.5773 | 1.7320 |
| Siliceous soils | dry | 39° | 0.8098 | 1.2349 |
| | moist | 44° | 0.9657 | 1.0355 |
| Vegetable soil | dry | 40° | 0.8391 | 1.1918 |
| | moist | 41° | 0.8693 | 1.1504 |
| Clayey soil | dry | 42° | 0.9004 | 1.1106 |
| | moist | 44° | 0.9657 | 1.0355 |
| Gravel | round | 30° | 0.5773 | 1.7320 |
| | sharp | 40° | 0.8391 | 1.1918 |

From this table it may be seen that in the same material the value of $\phi$ changes with the amount of moisture that the soil contains. Up to a certain limit, moisture tends to increase the value of $\phi$, while a larger quantity of moisture tends to decrease the angle of repose. In fact, when a fine soil contains a large quantity of water, its particles run on each other, and the angle of friction may decrease to very near zero, as in silts and muds, which flow almost like water when sufficiently wet.

5. **Cohesion in Soil.** — In a firm bank of earth the force of cohesion holds together the particles of the material, so that they oppose a certain resistance to a force tending to separate them from one another. Generally speaking, the force of cohesion can be considered as the resistance offered by the earth when being cut.

This cohesive force varies with the quality of the earth; it is very small in sand and siliceous soils, and greater in clay and similar materials. When earth is removed from its natural bed, its cohesion is entirely destroyed; when the soil is piled up again, it will regain its lost cohesion only in a

small degree unless well rammed and wetted. Consequently the cohesive force of a soil should be considered to be active only when the soil is in its natural position, whereas the force of friction always exists, whether the soil be in its original position or in an artificial embankment.

On account of the cohesive force existing in soils in their natural position, the slopes of trenches can be left more nearly vertical than those of embankments, since in constructing embankments the soil has been removed from its natural bed and consequently the force of cohesion destroyed. The slope of a cut is sustained by the combined action of the forces of cohesion and friction, while the slope of an embankment is sustained only by the force of internal friction. It is necessary, however, to remember that little reliance should be placed in the force of cohesion in calculating the slopes of cuts, since atmospheric influences tend greatly to alter it.

In opening a trench through loose soil, the sides of the excavation on account of the cohesion of the material can be left vertical up to a certain height. This height depends upon the quality of the soil. For each soil there is a maximum height at which it can remain vertical when cut; on an attempt to increase the depth of the cut beyond this, the sides of the trench will collapse. It is usual, therefore, to express the cohesive force of a soil in terms of the maximum height at which it can remain vertical when cut. Thus it is said that a soil stands vertically to a height of 4.75 ft.; this means that if it is attempted to increase the vertical cut to say $5\frac{1}{2}$ ft., the sides will collapse.

The value of the cohesive force of earths is generally expressed by the force which it is necessary to apply in order to destroy this force per unit of surface. For the sake of simplicity in the calculations the force of cohesion is expressed

in terms of the specific weight of the earth multiplied by the coefficient of cohesion. In other words, the coefficient of cohesion of a soil is the ratio of the cohesion per unit of surface to the weight per unit of volume. Calling this ratio $K$, the force of cohesion per unit of surface will be expressed by

$$C = K\gamma,$$

where $\gamma$ is the weight of the unit of volume of the soil considered.

In the following table are given the values of the force of cohesion in some soils as deduced from recent experiments:

| Quality of Soil | | Cohesion in Lbs. per Sq. Ft. |
|---|---|---|
| Ordinary earth | dry | 110.8 |
| | moist | 114.6 |
| Clayey soils | dry | 107.3 |
| | moist | 190.8 |

## Graphical Determination of Cohesion

**6.** The cohesive force in a bank of earth can be easily determined by graphical methods. Let a bank of earth be represented as in Fig. 3 by its profile $CBAG$; and let it be assumed that:

1. The mass of earth is homogeneous throughout;

2. The earth contains natural moisture;

3. The force of cohesion is uniform throughout the mass;

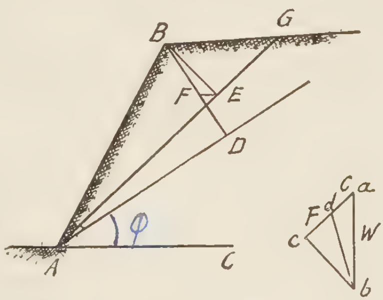



Fig. 3.

4. The stratification of the earth in the embankment does not affect the pressure due to the weight of the material;

5. The portion of the earth which tends to separate from the embankment will slide along a plane surface.

Properly speaking the last assumption is not correct, because the sliding surface is generally cylindrical, being formed by a straight horizontal line as generatrix moving along a cycloid as directrix. But in order to facilitate the calculations it is here supposed that the sliding surface is a plane. The results obtained in the use of this assumption are very close to those obtained by the theoretically exact method. Besides, the assumption is on the side of safety, as it neglects some of the resisting forces, while the destructive forces are all taken into consideration.

Then $ABCG$ being a bank of earth whose face has the slope $AB$, and $\phi$ being the angle of natural repose of the material: since the slope $AB$ makes with the horizontal line $AC$ an angle greater than $\phi$, the earth must be endowed with cohesion, for otherwise the slope $AB$ would coincide with the slope of repose. In order to compute the value of this force of cohesion which keeps in equilibrium the portion of the earth between the slope $AB$ and the slope of repose $AD$, it is supposed that the upper part of the mass of earth has a tendency to slide down along a plane represented in the figure by the line $AG$; then the prism $ABG$ will be under the action of four forces which are in equilibrium, as follows:

1. The weight $W$ of the prism $ABG$;

2. The reaction $N$ that the lower mass exerts upon the prism $ABG$, in a direction perpendicular to the plane of sliding;

3. The friction $F$ along the plane $AG$, depending upon the pressure $N'$ equal and opposite to the reaction $N$. This force of friction opposes the descending movement of the prism;

4. The resistance offered by the cohesion $C$ acting in the direction of $AG$ which tends to oppose the sliding of the prism.

Draw the force polygon: along a vertical line lay off a segment $ab$ equal to the weight $W$ of the sliding prism; from $a$ draw the line $ac$ parallel to $AG$ and from $b$ draw a normal to $AG$. Since the forces are in equilibrium the polygon $abc$ will be a closed one. The line $ac$ represents the forces which prevent the sliding of the prism, which are the friction and the cohesion. Therefore $ac$ must be equal to $F+C$. Suppose $d$ to be the dividing point between these two quantities, so that $cd$ represents the friction and $da$ the cohesion; join $b$ and $d$. The forces $F$ and $N$ are perpendicular to each other and their resultant $Q$ makes with $N$ an angle equal to the angle of repose $\phi$, or in other words the angle $cbd$ must be drawn equal to $\phi$.

Let the length of the embankment be equal to unity, *i.e.* let the depth of the prism normal to the plane of the paper be equal to 1. From the point $B$ drop the perpendicular $BE$ on $AG$; then the weight $W$ of the prism will be given by

$$W=\tfrac{1}{2}\gamma AG \times BE,$$

where $\gamma$ is the weight of the material per unit of volume.

To represent the forces by straight lines, it is necessary to assume a scale. Let in this case the scale of reduction for the forces be $\frac{1}{2}\gamma AG$ so that in the construction of the force polygon we shall have

$$W=BE.$$

Now, $AD$ being the natural slope, $BD$ a perpendicular from $B$ upon $AD$, and $EF$ an horizontal line drawn from $E$, the triangle $BEF$ will be equal to the triangle $dba$ of the

force polygon. For, $BE = W = ab$, $\angle BEF = \angle bac$ having the sides respectively perpendicular, $A \angle EBF = \angle abd$ (because $cba = GAC$ having their sides respectively perpendicular, and $\angle cbd = DAC = \phi$, hence $\angle dba = GAD = EBF$), which makes the triangles equal, having one side and two angles equal. Therefore $da = EF$. But in the force polygon

$$ca = cd + da = F + C,$$

and since the construction made $cd = F$, we have by subtraction

$$da = C, \text{ and consequently } C = EF.$$

Reducing from the assumed scale, we find that the force of cohesion, or the resistance due to the cohesion, is

$$C = \tfrac{1}{2}\gamma\, AG \times EF.$$

Since by assumption the force of cohesion is uniformly distributed, it acts at all points along the surface $AG$ and its value per unit of surface will be

$$\frac{C}{AG} = \tfrac{1}{2}\,\gamma EF.$$

By definition, the coefficient of cohesion is the quantity by which it is necessary to multiply the specific weight of the material in order to obtain the force of cohesion per unit of surface. But $\gamma$ being the unit of weight of the material, the coefficient of cohesion $K$ is $K = \frac{1}{2}\,EF$.

It is evident that the coefficient of cohesion $K$ is given in terms of the line $EF$, which varies with the different positions assumed by the probable planes of sliding. Consequently it will be interesting to determine what direction of $AG$ gives the greatest value of $K$.

Suppose the plane of sliding $AG$ to turn around the point $A$, Fig. 4. The line $EF$ is easily determined for each new

position of $AG$. The angle $BEA$ will always be a right angle; therefore the point $E$ will describe a circular arc described on $AB$ as diameter. This semicircle passes through $D$, as $\angle BDA$ is a right angle. Since $BD$ is a fixed line, because $AD$, the slope of repose, is fixed, it is evident that $EF$ will reach its greatest value when $E$ falls in the middle of the arc $BD$, or at $E'$. Consequently we have: maximum value of $K = \frac{1}{2} E'F'$, and this is obtained when $AG$ falls upon the line $AG'$ bisecting the angle $BAD$.

This means that the cohesion required to hold the mass from sliding reaches its greatest value when the probable plane of sliding bisects the angle between the slope $AB$ and the plane of natural repose of the material $AD$. The plane $AG$ is therefore to be considered as the most probable plane of sliding, the plane along which the prism of earth which tends to split off from the embankment would separate. In the considered embankment, if the force of cohesion of the earth is greater than $\frac{1}{2} E'F'\gamma$, the earth can be maintained under the slope $AB$, and $\frac{1}{2} \gamma E'F'$ will then represent the force of cohesion actually developed along the probable plane of rupture.

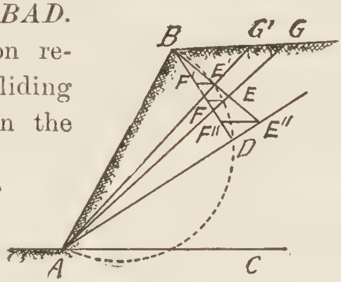



FIG. 4.

In Fig. 4, draw the line $BE'$ perpendicular to $AG'$ and produce it to $E''$ on the plane of natural repose; from this point $E''$ draw the horizontal line $E''F''$.

$$AE'' = AB \text{ and } E''F'' = 2\,E'F'.$$

For, since $E'$ is the middle point of the arc $BD$, the angle $BE'A$ is a right angle, as is also the angle $AE'E''$; and $AG'$ being the bisector of the angle $BAE''$, the triangles

$BAE'$ and $EAE''$ are equal, which gives $AB = AE''$, and $BE' = E'E''$, or $BE'' = 2\,BE'$. The triangles $BE'F'$ and $BE''F''$ are similar, having one angle common and the opposite sides parallel. Since the side $BE''$ is equal to twice $BE'$, and in similar triangles all sides are in the same proportion, we find that $E''F'' = 2\,E'F'$. Consequently the maximum value of the coefficient of cohesion $K$ will be

$$K = \tfrac{1}{2}\,E'F' = \tfrac{1}{4}\,E''F''.$$

The graphical construction for finding the coefficient of cohesion may be described as follows: from $B$ draw a perpendicular to $AD$, the line of natural slope; then make $AE'' = AB$ and from $E''$ draw $E''F''$ horizontally to intersect $BD$ in $F''$. The coefficient of cohesion $K$ is

$$K = \tfrac{1}{4}\,E''F''.$$

In the right-angled triangle $F''DE''$ the angle at $E'' = \phi$, because $CAE''$ and $AE''F''$ are alternative interior angles. Therefore $DE'' = E''F''\cos\phi$ and, since $K = \frac{1}{4}E''F''$, or $E''F'' = 4\,K$, we find,

$$DE'' = 4\,K\cos\phi.$$

Thus, the coefficient of cohesion $K$ can be expressed in terms of $DE''$,

$$K = \frac{DE''}{4\cos\phi}.$$

## The Parabola of Cohesion; Practical Applications

7. The preceding has dealt with the fact that an earth bank is in equilibrium so long as its side does not exceed a certain slope. We now will be concerned with the further observation that the greater the height of a bank or pile, the flatter will be the limiting slope. This observation may

be put more precisely by saying that, in a general way, the limiting slope is inversely proportional to the height of the bank.

It will be interesting to study the changing position of the top of the embankment $B$, as the embankment increases in height, supposing it to be maintained at the maximum slope. This will afford a simple way of determining the proper slope to be given to a bank which is to remain in equilibrium under the combined forces of friction and cohesion.

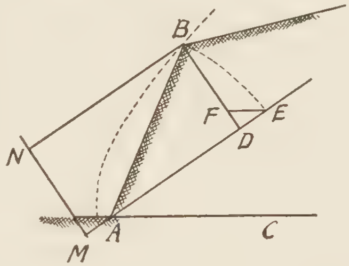


Fig. 5.

In a particular bank of earth the forces of friction and cohesion have a constant maximum value, *i.e.* the value of $\phi$ and $K$ are constant throughout the mass.

From $B$, Fig. 5, draw $BD$ perpendicular to the line of natural slope and take $AE = AB$; produce the line $AE$ and lay off a segment $AM = DE$; at $M$ erect a perpendicular to the $ME$ and from $B$ draw the line $BN$ parallel to the natural slope $AE$. It will result, then, that

$$BN = MD = EA = AB,$$

and

$$AM = DE = 4\ K \cos \phi,$$

which is a constant quantity because both $\phi$ and $K$ are constant. The straight line $MN$, therefore, has a fixed position. Since the line $BN$ is always equal to $AB$, it follows that the point $B$ is equidistant from the fixed point $A$ and the fixed line $MN$, no matter what height be assigned to the bank. This equality, it happens, is the character-

istic of a parabola. The locus of point $B$, the top of the bank of maximum slope, is therefore a parabola whose focus is at $A$ and whose directrix is the line $MN$; the slope of repose, $AD$, is the axis of the parabola.

The parabola of cohesion gives, for any given bank of earth, the various slopes of equilibrium corresponding to the different heights. Thus, for instance, in Fig. 6, draw from $A$ a perpendicular to the line $AC$; the point of intersection with the parabola will determine the height up to which a vertical face will remain in equilibrium. Again, the parabola of cohesion indicates that for heights less than this, the earth will stand even inclined forward from the perpendicular, a phenomenon which is illustrated along the edges of rivers and creeks, where the bank is undercut by the stream.

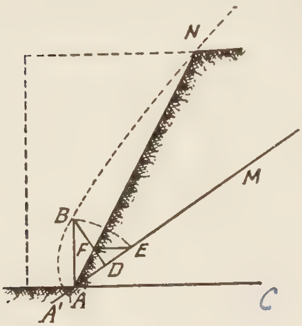


Fig. 6.

The following examples of the use of the parabola of cohesion may be serviceable to the student:

1. It is desired to know the slope which shall be given to a bank of earth 25 ft. high, when it is known that the material can stand with a vertical face up to a height of 7 ft.

From $A$, the foot of the embankment, Fig. 6, draw a vertical line and make $AB = 7$ ft., then draw the line $AM$ on the slope of repose, making with the horizontal an angle equal to $\phi$. On $AM$ lay off a segment $AE = AB$, from $B$ draw the line $BD$ perpendicular to $AM$, from $E$ the horizontal line $EF$; then $\frac{1}{4} EF = K$, the coefficient of cohesion. Produce $AM$ and lay off a segment $AA' = DE$ and describe

a parabola with vertex halfway between $A$ and $A'$, and with focus at $A$. Then drawing a horizontal line 25 ft. above the base, where this intersects the parabola, we find the top edge of the bank $N$. Connect $N$ with $A$; the slope $AN$ is the maximum slope of equilibrium corresponding to the height of 25 ft.

2. If instead of the vertical height we are given $K$, the coefficient of cohesion of the material, the solution is almost the same, the only difference being that the parabola is located by making the focal distance $A = A' = 4\,K \cos \phi$.

3. **The Stepped Slope.** — In high embankments instead of using a single slope it is usually preferred to cut the earth in successive small slopes alternate with benches. The parabola of cohesion gives an elegant solution of the problem of proportioning the successive slopes.

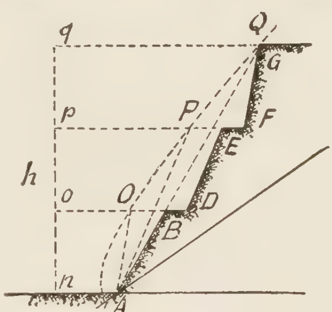


FIG. 7.

Let $h$, Fig. 7, be the total height of the embankment; $no$, $op$, $pq$, the height of the successive equal steps; and $\phi$ the angle of natural repose of the material.

Draw the parabola of cohesion as in any other case (the coefficient of cohesion or equivalent information being given, as in the preceding examples). From the points $o$, $p$, and $q$, draw horizontal lines, intersecting the parabola at $O$, $P$, $Q$, respectively. Connect $A$ with $Q$, then the line $AQ$ represents the slope of equilibrium for the total height $h$ of the embankment. Connect also $A$ with $O$ and $P$. The lines $AO$ and $AP$ will represent the slopes corresponding to one

and two steps respectively. Produce the line $oO$ to meet at $B$ the slope $AQ$ and lay off a segment $BD$ equal to the given width of the bench or berm. From $D$ draw $DE$ parallel to the slope $AP$, intersecting the line $pP$ produced at $E$, and again lay off a horizontal segment $EF$ equal to the required width of berm. Finally from $F$ draw a parallel to the slope $AO$ until it meets at $G$ the line $AQ$. The broken line $ABDEFG$ will be the required profile of the cut.

4. **The Slope of Equal Stability.** — The parabola of cohesion also affords an easy solution of the problem of determining in a graphical way the slope of equilibrium of equal stability for a bank of earth. The slope of equal stability differs from the slope of equilibrium corresponding to a certain height; while the latter has been considered as a straight line, the former is a curve. It is observed in trenches that have originally been cut vertical, and then were exposed to atmospheric influences for a long time, that the face assumes a curved outline; this is the slope or surface of equal stability.

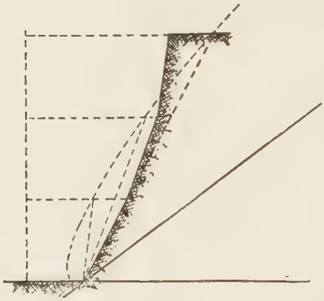


FIG. 8.

Suppose that in Fig. 7 the total height $h$ of the embankment was equal to 24 ft., and each one of the various sections $no$, $op$, $pq$ equal to 8 ft.; $AQ$ then is the proper slope for the total height of 24 ft., $AP$ for the height of 16 ft., and $AO$ for the height of 8 ft. The lowest step was given the slope proper to the height of 24 ft., the total height of the embankment, and thus was drawn the line $AB$. The remaining height from $B$ to the top of the embankment

is 16 ft., and for the second section was used the slope proper to the height of 16 ft., drawing *DE* parallel to *AP*. For the upper portion was used the slope proper to an embankment 8 ft. high by drawing *FG* parallel to *AO*.

If it were not for the berms, the various slopes would have formed a broken line, as indicated in Fig. 8. It is evident that the smaller the steps into which the total height of the embankment is divided, the more closely the broken line approximates to a curve. A division into seven steps is shown in Fig. 9. When the points are taken at an infinitesimally small distance from one another, the resultant slope of equilibrium, a slope of equal stability at all heights, becomes a smooth curve.

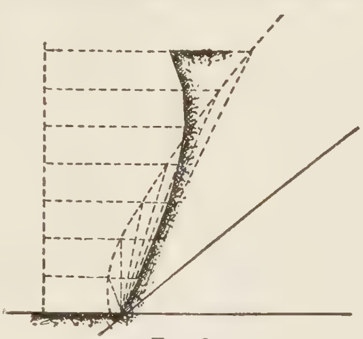


Fig. 9.

Once the slope proper to a given height of a bank of a given quality of soil having both friction and cohesion is known, it is then an easy matter to determine the safe slope for a bank in the same soil of any different height. Evidently, also, if it is desired to determine the slope which will call into play only $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ of the total force of cohesion, so that the bank is designed with a factor of safety of 2, 3, 4, etc., in respect to its force of cohesion, it is only necessary to find the slope of equilibrium or the curve of equal stability for a coefficient of cohesion taken at $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ of the actual value. For this purpose the parabola of cohesion is employed in the same manner as in the preceding cases; the focal distance, however, instead of being taken equal to $4\,K \cos \phi$, is made $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ of this value.

## Analytical Calculation of Cohesion

**8. Slope without Surcharge.** — The influence of the cohesion can be calculated also by analytical methods. Consider an embankment under the slope $AB$, and denote by

$\phi$ the angle of repose,

$\beta$ the angle included between the slope $AB$ and the plane of natural repose represented by the line $AD$,

$l$ the length of the slope $AB$,

$h$ the vertical height of the embankment, and

$K$ the coefficient of cohesion.

From the preceding section we know that, in Fig. 5,

$$AB = AE = AD + DE,$$

but $$DE = 4K \cos \phi \text{ and } AD = AB \cos \beta = l \cos \beta.$$

Then the length of the slope of the embankment will be

$$l = l \cos \beta + 4K \cos \phi,$$

from which is deduced the value of the coefficient of cohesion,

$$K = \frac{l(1 - \cos \beta)}{4 \cos \phi}; \qquad (a)$$

but $$\cos \beta = \cos^2 \frac{\beta}{2} - \sin^2 \frac{\beta}{2},$$

and $$l - \cos \beta = l - \cos^2 \frac{\beta}{2} + \sin^2 \frac{\beta}{2};$$

and since $$l - \cos^2 \frac{\beta}{2} = \sin^2 \frac{\beta}{2},$$

we shall have $$l - \cos \beta = 2 \sin^2 \frac{\beta}{2}.$$

Substituting this value in the formula ($a$), we have

$$K = l\frac{\sin^2\frac{\beta}{2}}{2\cos\phi}. \qquad (b)$$

Thus the coefficient of cohesion $K$ is given by the length of the slope of the embankment multiplied by the square of the sine of half the angle between the slope and the plane of natural repose, divided by twice the cosine of the angle of repose $\phi$.

The coefficient of cohesion can be found also in terms of the height of the embankment. From Fig. 5 it is easily seen that $h = l\cos\partial$, where

$$\partial = 90 - (\beta + \phi); \text{ then } \cos\partial = \sin(\beta + \phi)$$

and
$$l = \frac{h}{\sin(\beta + \phi)}.$$

Substituting this value in equation ($b$), we shall have the coefficient of cohesion $K$ in terms of the height of the embankment and both the angles $\phi$ and $\beta$:

$$K = \frac{h\sin^2\frac{\beta}{2}}{2\sin(\beta + \phi)\cos\phi}, \qquad (c)$$

or
$$h = 2K\frac{\sin(\beta + \phi)\cos\phi}{\sin^2\frac{\beta}{2}}. \qquad (d)$$

The equations ($c$) and ($d$) involve only the four quantities $K$, $h$, $\beta$, and $\phi$, so that when any three of them are given the fourth may be calculated.

The force of cohesion is often given in terms of the maximum height at which the earth will stand with vertical face.

For a vertical face, $\beta + \phi = 90°$. Call $h'$ the value of $l$, or the height at which the embankment remains vertical due to the cohesion. Observe that

$$\cos\phi = \sin(90 - \phi) = 2\sin\frac{(90-\phi)}{2}\cos\frac{90-\phi}{2}.$$

Substitute these values in equation ($b$); then

$$K = \frac{h'}{4}\frac{\sin^2\frac{90-\phi}{2}}{\sin\left(\frac{90-\phi}{2}\right)\cos\left(\frac{90-\phi}{2}\right)} = \frac{h'}{4}\tan\left(45 - \frac{\phi}{2}\right), \qquad (e)$$

from which is deduced

$$h' = 4K\cot\left(45 - \frac{\phi}{2}\right) = 4K\tan\left(45 + \frac{\phi}{2}\right). \qquad (f)$$

Putting in equation ($c$) the value of $K$ given by equation ($f$), we obtain the value of $h$ in terms of $h'$:

$$h = \frac{h'}{2}\frac{\sin(\beta+\phi)\cos\phi}{\tan\left(45 - \frac{\phi}{2}\right)\sin^2\frac{\beta}{2}}. \qquad (g)$$

This formula gives the value of $h$ when $h'$, $\beta$, and $\phi$ are given; or it can be used for calculating the angle $\beta + \phi$ to be given to a slope when $h'$, $\phi$, and $h$ are known.

**9. Surcharged Slope.** — Consider a mass of earth limited above by a surface $BD$ which makes an angle $\alpha$ with the horizontal line passing through $B$. The embankment will then be surcharged by an additional weight $W$ per unit of surface, and the limiting height for a given slope $\beta + \phi$ will be decreased.

In Fig. 10, $AD$ represents the plane of sliding of the prism $ABD$, $W$ its weight with surcharge included, and

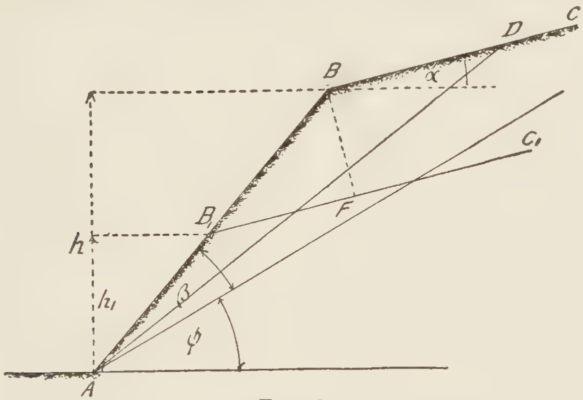



FIG. 10.

$\gamma$ the unit of weight of the earth. The total weight of the prism is

$$W = \gamma \frac{AB \times BD}{2} \sin(\beta + \phi - \alpha) + W \times BD,$$

or making $\gamma = 1$ and factoring,

$$W = \frac{AB \times BD}{2} \sin(\beta + \phi - \alpha)\left(1 + \frac{2\ W}{AB \sin(\beta + \phi - \alpha)}\right).$$

But

$$AB = \frac{h}{\cos(90 - \beta - \phi)} = \frac{h}{\sin(\beta + \phi)},$$

and substituting this,

$$W = \frac{AB \times BD}{2} \sin(\beta + \phi - \alpha)\left(1 + \frac{2\ W \sin(\beta + \phi)}{h \sin(\beta + \phi - \alpha)}\right),$$

which means that the weight of the sliding prism increased by the surcharge by a quantity $\dfrac{2\ W \sin(\beta + \phi)}{h \sin(\beta + \phi - \alpha)}$, the maxi-

mum height $h$ at which the bank will stand can be deduced from equation ($d$) corrected by this factor, thus:

$$\frac{2\,K \sin(\beta+\phi)\cos\phi}{\sin^2\frac{\beta}{2}} = h\left(1+\frac{2\,W\sin(\beta+\phi)}{\sin(\beta+\phi-\alpha)}\right),$$

from which is deduced the value of $h$,

$$h = 2\,K\frac{\sin(\beta+\phi)\cos\phi}{\sin^2\frac{\beta}{2}} - \frac{2\,W\sin(\beta+\phi)}{\sin(\beta+\phi-\alpha)}.$$

If an earth embankment can remain vertical to the height $h_1$, and then is surcharged with an equally distributed weight $W$ per unit of surface, the height at which it will now stand vertical is decreased in the ratio $\frac{2\,W\sin(\beta+\phi)}{\sin(\beta+\phi-\alpha)}$.

This factor can be marked directly on the slope by making $BB' = \frac{2\,W}{\sin(\beta+\phi-\alpha)}$, which is obtained by drawing a line $B_1C_1$ parallel to $BC$ at a distance $BF$ from $B$ equal to $2\,W$. When the surface above $BC$ is horizontal, $\alpha = 0$ and $BF = 2\,W$ gives directly the decreased height of the slope.

## Earth Slopes in Practice

**10.** In public works it is a common practice to use the slope of 1 to 1 for cuts and the slope of $1\frac{1}{2}$ to 1 for fills. The convenience of such a general rule is founded upon the fact that it is impracticable to calculate the proper slopes for the various cuts and fills encountered in any ordinary piece of work. Since the character of the soil changes continuously along the line of the work, it would be a very slow and expensive matter to calculate the proper slopes; and the

resulting delays and the increased cost of work would not bring any material benefit. For this reason short practical rules are always followed. But the convenience of the practical rules should not mislead the engineer when he has to deal with a special problem, as, for instance, making a deep cutting for the approaches of a tunnel in railroad work, or for a canal. In these and similar cases, giving to the cuts their proper slopes will result in great savings both in the original cost of construction and in the cost of maintenance.

The slope of $1\frac{1}{2}$ to 1 commonly given to fills is a safe slope in most cases. The earth in a fill stands up only by virtue of the force of friction, since the force of cohesion has been destroyed in the removal of the soil from its natural bed. From the table given on p. 5 it is seen that the natural slope of most of the common soils is smaller than 1.5. Hence the slope of $1\frac{1}{2}$ to 1 as generally used may be considered safe. But there are soils, as rounded gravel, for instance, that have a natural slope as high as 1.7 to 1; consequently, when such materials are used in forming an embankment, a slope of $1\frac{3}{4}$ to 1 should be given instead of the usual slope of $1\frac{1}{2}$ to 1.

The slope of $1\frac{1}{2}$ to 1 given to a certain embankment may be considered a safe slope, when the material is dry or contains only a small percentage of water; but when the soil contains a large quantity of water it may assume a smaller slope. Thus, in the same table, p. 5, the natural slope for sand when dry is given as 1.43 to 1, when moist as 1.19 to 1, when very wet, 1.73 to 1. From this it can be seen that in determining the slope to be given to an embankment, chiefly composed of sand, it is safer to give a slope of 2 to 1, instead of the usual slope, unless local conditions insure the prompt discharge of water and make it practically impossible for the embankment to become saturated with water.

It is in the cuts that practical rules should be applied most cautiously. If a bank of earth is left with the slope of 1 to 1, while the angle of repose of all loose materials is smaller than 45°, it follows that the engineer takes advantage of the cohesive force of the soil to maintain the bank. From the parabola of cohesion we learn that the higher the bank of earth, the smaller is its slope of equilibrium. It is evident that it would be more economical to cut the sides of the bank to a compound or curved slope, closely following the curve of equal stability, instead of giving to the total height of the embankment a single slope, since the latter involves a larger amount of excavation. It is not necessary that the slope of equal stability be designed for the earth in equilibrium; it may be designed with a certain factor of safety in respect to the cohesive power of the soil, this factor of safety being assumed according to the local conditions, as will be discussed farther on.

Using for deep trenches side slopes following the curve of equal stability, designed with a factor of safety of 2, for instance, only one half the available force of cohesion is brought into play all through the mass of the bank of earth, and consequently also at the bottom of the bank. In the practical slope of 1 to 1 there is a very large factor of safety in the upper part of the bank, which means that there is a large amount of useless excavation done on the upper section of the cut, while near the foot of the bank the force of cohesion is taken almost at the equilibrium point. The foot of the bank is therefore the most dangerous point, the one most liable to a collapse. In actual work it is the lower portion of the bank that always collapses first, and consequently the smallest slope should be given to the lower part of the cut.

This is obtained only by designing the slope according to the curve of equal stability.

Figure 11 represents the various slopes of equal stability and the practical slope of 1 to 1, for a bank of earth 50 ft. high, in which the coefficient of cohesion was assumed to be equal to 1.3, and the angle of natural repose = 30°. The line *AB* indicates the slope of 1 to 1; the slope of equal stability designed with the value of cohesion taken at the equilibrium point is indicated by *AC*, while *AD* and *AE*

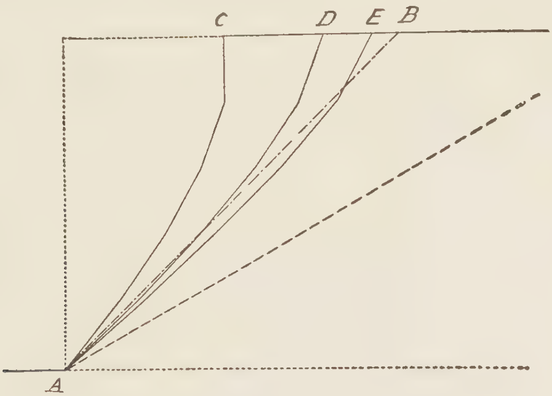



FIG. 11.

represent the slope designed with factors of safety 2 and 3 respectively. Comparing these various slopes it is seen that the slope of equilibrium forms an angle of 50° with the horizontal at the lowest portion of the cut, while the slope of equal stability designed with a factor of safety of 2 forms an angle of 43° 15′, and the slope designed with a factor of safety of 3, an angle of 40° 15′.

The amount of excavation back of the vertical for a bank 1 ft. in length will be 1250 cu. ft. for the slope of 1 to 1, 873.5 cu. ft. for the slope of equilibrium, 1165.5 for a slope with fac-

tor of safety 2, and 1260 cu. ft. for the slope with a factor of safety 3. From this it is seen that the usual slope of 1 to 1 requires almost the same amount of excavation as a slope designed with a factor of safety equal to 3; while the former has an angle of 45°, the second forms with the horizontal an angle of 40° 15′. A curve of equal stability in which the lowest portion makes an angle of 45° with the horizontal would require only 950 cu. ft. of excavation, a saving of nearly $\frac{1}{3}$ the total amount.

If such economy can be obtained in cuts of only 50 ft., it is easy to imagine the enormous amount of excavation which could be saved by using the curve of equal stability in cutting very deep trenches, as for instance at the great cuts of the Panama Canal, 300 ft. deep and extending several miles in length.

Owing to the fact that in deep trenches heterogeneous material is encountered as a rule, it would be very difficult to determine the force of cohesion of the various materials and integrate the various slopes of equal stability in a single curve best suited to the particular case under consideration. But the engineer should generally adopt a slope proper to the material endowed with the smallest cohesive force. In doing this he must not forget that the lower strata of the bank are most liable to collapse, so that when a very loose material is met with in the upper strata, the cut may be made with a slope greater than if the same material were encountered farther down.

Since the force of cohesion in any embankment is greatly altered by atmospheric influences, it is safest to rely only on a fraction of it, or in other words to use the force of cohesion with a certain factor of safety, 2 or 3, according to local conditions. A factor of safety of 2 would give sufficient secur-

ity under the climatic conditions of this country, while a factor of safety of at least 3 should be used in countries, like Panama, situated in the tropical regions where heavy rains occur in some part of the year so frequently as to entirely permeate the soil.

# CHAPTER II

## RETAINING WALLS

**11. Theories of Earth Pressure.** — Retaining walls are erected for the purpose of supporting earth embankments or the sides of cuts, at slopes steeper than would be in equilibrium without artificial support. In a bank which is held up by a retaining wall, the internal forces (friction and cohesion) are not sufficient to keep the face from caving. Consequently the bank presses against the wall with a certain intensity of pressure. The proportions of the wall will, of course, be determined by the amount of this pressure, so that in order to design a retaining wall, the pressure which the structure has to resist must first be calculated.

Although many authors have elaborated theories of earth pressure against walls, all these theories can be classified in two groups: (1) the theory of the sliding prism, and (2) the analytical theory.

The theory of the sliding prism originated with Vauban, a general in the French army in 1687. General Vauban assumed that the natural slope is constant for all soils and equals 45°, and that the triangular prism of soil above the plane of repose, sliding on that plane, causes a thrust equal to the actual thrust against the retaining wall. He resolved the weight of this sliding prism into components normal and parallel to the natural slope, and considered the second component as the real pressure against the wall.

Vauban's theory was improved by Belidor in 1729. Belidor considered that the pressure against a wall is not equal to the tangential component of the sliding prism, as stated by Vauban, but is less than this component by a certain amount due to the friction of the sliding prism on the soil beneath. Finally, Captain Coulomb of the French army, in the year 1773, still further developed the theory of the sliding prism. He resolved the reaction of the wall (equal and opposite to the pressure) and the weight of the sliding prism into their components parallel and normal to the plane of rupture. He made the difference of the components parallel to the plane of rupture equal to the frictional resistance opposed by the mass of earth below the sliding prism, and from this equation deduced the value of the total pressure against the wall. This theory, later perfected by Prony, François, and Poncelet, is the one most commonly used; it is generally known as Coulomb's theory of retaining walls, or the theory of the sliding prism.

In the year 1856, Professor W. M. Rankine deduced the value of the pressure of the earth against a retaining wall by means of a new and more scientific principle. He investigated the conditions of equilibrium of an interior element in a homogeneous mass of earth deprived of cohesion and unlimited in every direction, and thus determined the pressures erected upon the bounding surfaces of the element. From this, when a plane surface limiting a mass of earth was given, the direction, magnitude, and position of the total pressure upon this plane surface could be easily determined. Such an analytical method was followed by Moseley, Winkler, Levy, Considère, and Weyrauch. However, it has remained largely in the field of scientific investigations and

has not been so extensively used for practical purposes as the theory of the sliding prism.

In this and the following sections is given a method for determining earth pressure graphically, according to the method of Professor Rebhann, based upon the principle of the sliding prism.

### Graphical Method after Rebhann

**12.** Let a bank of earth held up by a retaining wall be represented by its profile *ABDC*, Fig. 12, and let the following assumption be made:

1. The pressure against the wall is caused by a prism of earth which tends to separate from the bank and slide along a plane surface. Such an assumption is not strictly accurate, because the surface of sliding or surface of rupture is not a plane but a cylindrical surface, having a cycloid as directrix. But by assuming the surface of sliding to be a plane surface, we greatly simplify calculations and the results are very close to the actual facts, erring a little on the safe side (since the sliding prism as assumed is a little larger than the mass which actually tends to split off).

2. The specific weight of the material be uniform all through the mass of earth forming the embankment. This assumption also is not strictly correct, because the specific weight varies with depth, humidity, etc.; but the laws of the variation are still unknown and the error involved in considering the embankment to be of constant specific weight is so small as to be negligible in practice.

3. The earth is devoid of cohesion. Usually the earth behind a retaining wall is filled in after the wall is built, so that in most cases the cohesive force has been entirely destroyed.

In Fig. 12, $AB$ is the back of a retaining wall supporting a mass of earth limited above by a cylindrical surface whose profile is represented by an irregular line as shown.

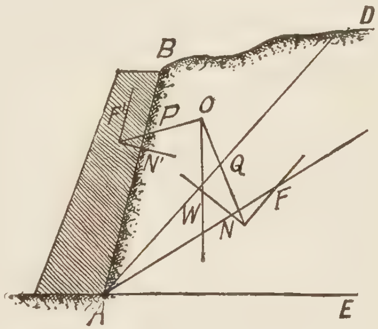


Fig. 12.

Since it is assumed that the earth is entirely devoid of cohesion, the prism of earth above the plane of natural repose would have a tendency to separate from the mass, and slide along the plane of repose. But in order to slide, the prism would also slide along the face $AB$, the back of the wall, and it thereby develops an upwardly directed force of friction which will counteract in part the falling tendency of the prism. In consequence, instead of sliding along the plane of natural slope it would slide along the plane $AD$, making with the horizontal $AE$ an angle greater than the angle of natural repose $\phi$.

Let $P =$ the pressure against the wall,

$\phi =$ the angle of natural repose of the earth,

and $\phi' =$ the angle of friction between earth and the masonry of the wall.

Then the forces acting upon the prism $ABD$ are: the weight $W$ of the prism; the reaction $N$ of the lower mass of earth, normal to the plane of sliding; the friction $F$ produced by the normal pressure $N$; the reaction $N'$ normal to $AB$, which the wall opposes to the prism; and the friction $F'$ between wall and earth, depending upon $N'$.

We know that

$$F = N \tan \phi,$$

since $\phi$ is the angle of friction between any two portions of the earth; also

$$F' = N' \tan \phi'.$$

The resultants $P$ and $Q$ of the pair of forces $F'$, $N'$, and $F$, $N$, are

$$Q = \sqrt{F^2 + N^2}$$

and

$$P = \sqrt{F'^2 + N'^2}.$$

Substituting for $F$ and $F'$ their values, we have

$$Q = N\sqrt{1 + \tan^2 \phi}$$

and

$$P = N'\sqrt{1 + \tan^2 \phi'}.$$

But

$$1 + \tan^2 \phi = \sec^2 \phi = \frac{1}{\cos^2 \phi}$$

and

$$1 + \tan^2 \phi' = \sec^2 \phi' = \frac{1}{\cos^2 \phi'},$$

so that

$$Q = N\sqrt{\frac{1}{\cos^2 \phi}}$$

and

$$P = N'\sqrt{\frac{1}{\cos^2 \phi'}},$$

or

$$Q = \frac{N}{\cos \phi} \text{ and } P = \frac{N'}{\cos \phi'}.$$

This means that the prism $ABD$ is in equilibrium when the angles made by the resultants $Q$ and $P$ with the normals are respectively equal to $\phi$ and $\phi'$, and intersect the line of action of the weight of the prism in the same point $O$.

We may deduce the general rule that the pressure $P$ of the earth against the back of the wall is never normal to the

back of the wall, but has a downward slope, making with the normal an angle $\phi'$ equal to the angle of friction between the earth and the wall.

**13. Values of $\phi$ and $\phi'$.** — It is very easy to find by experiment the value of $\phi$ for different soils, but it is not so easy to find the value of $\phi'$ because we have no satisfactory method of measuring it, and because it varies greatly with the different kinds of earth and masonry.

Colonel Andé assumes $\phi' = 26° 34'$; Poncelet and Moseley have given to $\phi'$ the values of $27° 2'$, $18° 47'$, and $21° 48'$; Scheffer, Rebhann, Otto, and Curioni assume $\phi' = \phi$; while still others make $\phi' = \frac{1}{3}\phi$. The older authors, as Coulomb, Prony, and others, did not consider $\phi'$ at all, thus making it equal to zero. Following Professor Rebhann we shall assume $\phi' = \phi$. The surface of the back of the wall never is smooth, but rather is very rough. When the earth backing is rammed against this rough surface, it fills all the cavities and recesses in the masonry. If sliding took place, evidently the earth could not disengage itself from the irregularities in the masonry, but rather a layer of earth just along the masonry would stay in place, and the prism would slide along this layer of earth. This condition would be a sliding of earth on earth, for which the friction factor is $\phi$. It seems proper, therefore, to assume that the friction between earth and back of wall be the coefficient $\phi' = \phi$.

**14. Location of Plane of Rupture.** — Consider an embankment in equilibrium or just before the caving of its face; suppose its length (in Fig. 13 perpendicular to the drawing) is 1.

Let $AD$ represent the plane of rupture. We will denote by $\beta$ the angle that the plane of rupture makes with the

natural slope. The prism $BAD$, sliding along the planes $AD$ and $AB$, acts as a wedge, developing the pressures $P$ and $Q$, whose directions depend upon the corresponding angles $\phi$ and $\phi'$, and whose intensities depend upon the weight $W$ of the prism. We shall limit our investigation to the forces $W$, $Q$, and $P$. Their directions are first to be found, for when these are known, the numerical amount of $Q$ and $P$ is easily calculated, thus:

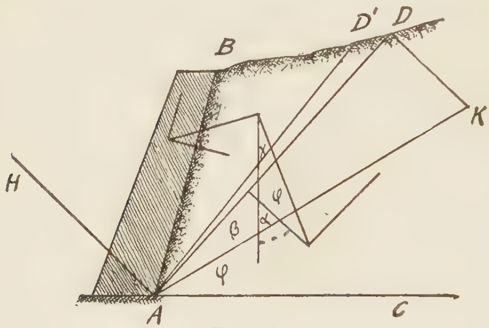


Fig. 13.

Draw a vertical line (Fig. 14) of length $ab = W$; from $a$ draw a parallel to $P$, and from $b$ a parallel to $Q$. Then the triangle $abc$ is the triangle of forces, $bc$ representing the value of $Q$ and $ac$ the value of $P$.

It remains to fix the plane of rupture $AD$, in order to determine the directions of $P$ and $Q$. Suppose the plane $AD$ turns around $A$ by a very small angle $d\beta$; represent its new position by $AD'$ ($d\beta$ is exaggerated in Fig. 14). The weight $W$ of the prism will be diminished by a very small quantity $dW$, represented by $bb'$ in the triangle of forces (Fig. 14). Professor Rebhann, to whom this demonstration is due, assumes that upon slightly varying the plane of rupture the pressure $P$ remains constant and only $Q$ varies, its new value being $Q'$; $Q'$ is represented by $cb'$ in the triangle of forces (Fig. 14), the angle $bcb'$ being equal to $d\beta$.

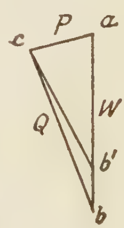


Fig. 14.

In Fig. 13 the angle which $Q$ makes with $N$ (the normal to the plane of rupture) is equal to $\phi$, the angle of internal friction of the soil, equal to the angle of repose. Now the angle $a = \gamma + \phi$, being an exterior angle of the triangle. Also $a = \phi + \beta$, as the angles $a$ and the angle $\phi + \beta$ have their sides respectively perpendiculars. Hence we have

$$\gamma = \beta.$$

Turning the plane of rupture around $A$ by an angle $d\beta$, the resultant $Q$ is turned through a corresponding angle $d\gamma$, and for the same reason as above we will have

$$\beta + d\beta = \gamma + d\gamma,$$

and since $$\beta = \gamma,$$

therefore $$d\beta = d\gamma.$$

But the angles $d\gamma$ and $bcb'$ are equal, having their sides respectively parallel; in consequence

$$d\beta = bcb'.$$

In the construction of the triangle of forces we may use such a scale that $Q$ be represented by the line

$$bc = AD.$$

In this case

$$\triangle\, bcb' = \triangle\, ADD',$$

and having already assumed that

$$W = ab \text{ and } bb' = dW,$$

we will have

$$\text{area } ABD : \text{area } ADD' = ab : bb',$$

because the weights of prisms of the same material and equal height are to each other as the areas of their bases.

Further, from the triangle of forces we have

$$\text{area } abc : \text{area } bcb' = ab : bb'.$$

Combining the two proportions gives the proportion

$$\text{area } BDA : \text{area } \cancel{K}ADD' = \text{area } abc : \text{area } bcb'.$$

Since we have drawn the triangle of forces to such a scale

$$\text{area } ADD' = \text{area } bcb',$$

it will result that

$$\text{area } BAD = \text{area } abc.$$

Therefore $Q$ makes with the normal to $AD$ an angle $\phi$ and with the vertical $W$ an angle $DAC - \phi = \beta$.

If we apply the triangle of forces to the cross section in Fig. 13, in such a position that $b$ coincides with $A$, and if we then turn it through an angle $90 - \phi$, the line $bc = AD$ will coincide with $AD$ and $ba$ with $AK$; the line $ca$, which in the original position made an angle $90 - \phi'$ with the back of the wall, after rotation will make an angle

$$90° - \phi + 90° - \phi' = 180° - (\phi + \phi'),$$

and consequently the line $ac$ will fall upon $DK$. Its direction is parallel to the line $AH$ drawn from $A$ at an angle $\phi + \phi'$ with the back of the wall $AB$.

The triangle $ADK$ will then be equal to the triangle $abc$. Since we already know that

$$\text{area } ABD = \text{area } abc,$$

it follows that

$$\text{area } ABD = \text{area } ADK.$$

That is, the plane of rupture $AD$ divides the surface $ABDK$ into two equal areas.

The line $AH$ parallel to $DK$ and at angle $\phi + \phi'$ with the back of the wall is called the directrix.

**15. The Triangle of Pressure.** — From the triangle of forces we have

$$W : Q : P = ab : bc : ac = AK : AD : DK. \qquad (1)$$

From these proportions we 'may reduce the following principle:

The weight of the sliding prism, the reaction of the plane of rupture, and the resistance opposed by the back of the wall are proportionals to the three sides of the triangle $ADK$.

In Fig. 15, with center in $A$ and radius $AD$, draw an arc

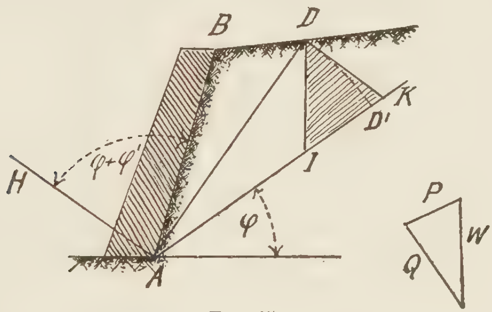



FIG. 15.

$DD'$, so that $AD' = AD$; and with center at $K$ mark off $KI = DK$. Draw $DI$, and $DD''$ then will have

$$AK : AD'' : KI = \triangle ADK : \triangle ADD'' : \triangle DKI,$$

or, which is the same,

$$AK : AD : DK = \triangle ADK : \triangle ADD'' : \triangle DKI.$$

From equation (1) we have

$$AK : AD : DK = W : Q : P;$$

substituting these values gives

$$W : Q : P = \triangle ADK : \triangle ADD'' : \triangle DKI,$$

and as the area of $\triangle ADK = \triangle ABD$ is proportional to the weight $W$, the triangles $ADD''$ and $DKI$ will be respectively proportional to $Q$ and $P$.

The reaction $Q$ of the plane of rupture is equal to the weight of a prism having $ADD''$ for base ; or, denoting by $\gamma$ the weight of a cubic unit of the soil,

$$Q = \triangle ADD'' \times \gamma.$$

The pressure $P$ is equal to the weight of a prism of unit height having the triangle $DKI$ as its bases,
or

$$P = \triangle DKI \times \gamma.$$

The triangle $DKI$ is called the triangle of pressure.

The demonstration last given (Fig. 15) is due to Professor Weyrauch.

## Application of the Method to Various Practical Cases

**16.** In order to utilize the theory just given in solving practical problems, we will study the graphical construction for determining both the plane of sliding and the triangle of pressure in various embankments :

1. When the embankment is surcharged, its upper surface being inclined at any angle.

2. When the embankment has no surcharge, its upper surface being horizontal.

3. When the surcharge is maximum, its upper surface sloping at the angle of repose of the soil.

4. When the upper surface of the embankment is a broken surface, composed of several planes so that its profile forms a broken line.

5. When the upper surface of the embankment is a cylindrical surface, whose profile is a curve.

The angle which the upper surface of the embankment makes with the horizontal will here be called the angle of surcharge.

17. CASE I. **Surcharged Embankment; Any Angle of Surcharge.** — Let $ABC$, Fig. 16, be the cross section of the

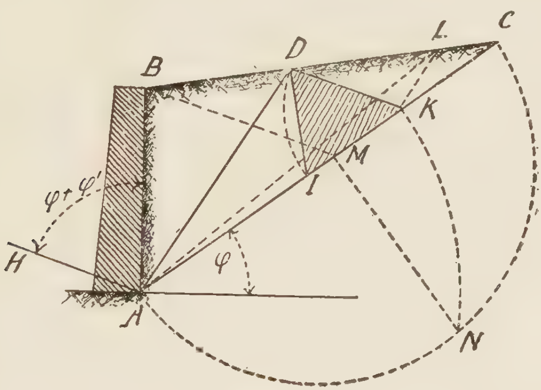



FIG. 16.

bank of earth limited above by the plane $BC$, $AB$ being the back of the retaining wall.

Supposing the problem to have been already solved, $AD$ will be the plane limiting the prism causing the pressure. From $A$ draw the directrix $AH$, making with the back of the wall $AB$ an angle equal to $\phi + \phi'$. From $D$ draw $DK$ parallel to $AH$; from $K$ draw $KL$ parallel to $AD$; and from $B$ draw the line $BM$ parallel to $AH$.

The two triangles $ADK$ and $ADL$ are equivalent because they have the same base and are included between parallels. Also, since $\triangle\ ABD = \triangle\ ADK$ (because the plane of rupture divides the area $ABDK$ into two equivalent triangles), the triangles $ABD$ and $ADL$ must be equivalent. Considering

$BD$ and $DL$ as the bases of these triangles, they have the same altitude (a normal from $A$ to $BL$), and hence their bases must be equal, or $BD = DL$.

In the similar triangles $ADC$ and $CLK$,

$$AK : AC = DL : DC = BD : DC.$$

Also, in the similar triangles $BMC$ and $DKC$,

$$BD : DC = MK : KC.$$

Therefore,

$$AK : AC = MK : KC.$$

The two latter quantities may be written in terms of the former as follows:

$$MK = AK - AM \text{ and } KC = AC - AK,$$

which substituted in the previous equation give

$$AK : AC = AK - AM : AC - AK,$$

which proportion, multiplied out, gives

$$\overline{AK}^2 = AM \times AC.$$

This means that $AK$ is a mean proportional between the segments $AM$ and $AC$. Therefore by drawing a semicircle on $AC$ as diameter, erecting a perpendicular $MN$ at $N$, and marking off $AK =$ chord $AN$, we determine the point $K$. For the triangle $ANC$ is a right triangle, being inscribed in a semicircle; and in a right triangle either leg is a mean proportional between the adjacent segment of the hypotenuse cut off by a perpendicular from the vertex to the hypotenuse and the whole hypotenuse.

This gives a simple means for locating the plane of rupture. $AC$ being the plane of repose, draw $BM$ parallel to the directrix $AH$. When drawing the semicircle $ANC$

locate $K$ as described; draw $KD$ parallel to the directrix $AH$. Draw $AD$ which is the plane of rupture.

To describe the triangle of pressure: with center at $K$ and radius $KD$, lay off $KI = KD$; unite $I$ with $D$. The triangle $KDI$ will be the triangle of pressure or the triangle whose area multiplied by the unit of weight of the material gives in pounds the total pressure against the retaining wall, per lineal foot of wall.

Point $K$ may also be located as shown in Fig. 17: describe a semicircumference on $MC$ as diameter, and from $A$

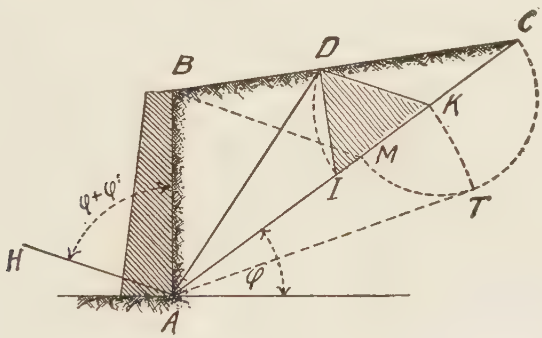



FIG. 17.

draw the tangent $AT$. Revolving $AT$ about $A$ into $AC$ gives $AK$. This method is based on the theorem of geometry that the tangent to a circle is a mean proportional between the whole secant and its internal segment.

Similar methods of construction leading to exactly the same result can be made with the use of other lines as base in place of $AC$:

(*a*) The upper surface of the embankment $BC$.

(*b*) The back of the retaining wall $AB$.

(*c*) The directrix $AH$.

(*a*) In Fig. 18 project the points $A$, $M$, and $C$ upon $BC$ or $BC$ produced by lines parallel to the directrix, which gives the points $H$, $B$, and $C$. Since the equation of Fig. 16, viz.,

$$\overline{AK}^2 = AM \times AC,$$

is true also for any set of projections of these lines, the construction above applied to $AC$ and $AM$ may be applied

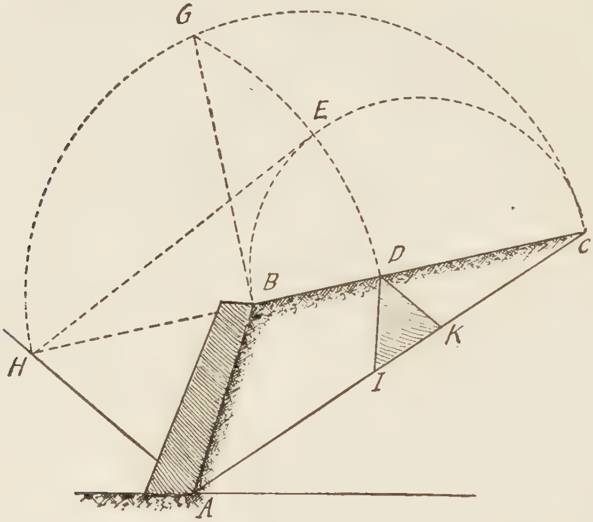



FIG. 18.

directly to $HC$ and $HB$, giving $HD$ according to the equation,

$$\overline{HD}^2 = HB \times HC.$$

This gives a point which is the projection of $K$ by a line parallel to $AH$, and as will be seen by referring to Fig. 15 this is the point $D$ in the plane of rupture. Therefore the construction of Fig. 18 gives point $D$ and defines the plane of rupture directly. The details of the construction are shown in Fig. 18.

Having $D$, draw $DK$ parallel to $AH$ to intersect the surface of repose $AC$ at $K$. Then lay off $KI = KD$. Triangle $DKI$ is the triangle of pressure.

In Fig. 18, for convenience, the value of $\phi$ has been taken larger than in former cases and the resulting triangle of pressure is smaller.

The construction of Fig. 17 is applicable here also, of course, as shown in Fig. 18. Describe a semicircle on $BC$ as diameter, draw a tangent from $H$, and revolve $HE$ to $HD$.

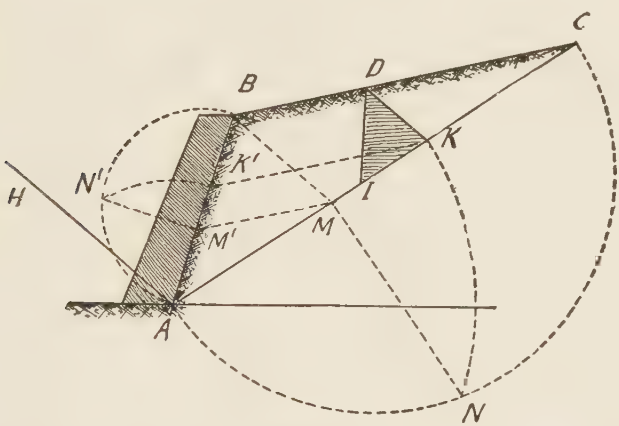



FIG. 19.

($b$) Project the points $A$, $M$, and $C$, Fig. 19, on the back of the wall $AB$ by lines parallel to $BC$, giving the points $A$, $M'$, and $B$. As in the preceding case, the construction for a mean proportional is applied to $AB$ and $AM'$, giving the point $K'$, which projected by a line parallel to $BC$ gives the point $K$.

The construction is clearly indicated in Fig. 19. Point $M$ is found as before by projecting $B$ upon the surface of repose $AC$ by a line $BM$ parallel to the directrix.

(*c*) Referring to Fig. 20, project the points $H$, $B$, and $C$ on the directrix $AH$ by lines parallel to $AC$, which give points $H$, $B'$, and $A$. Then $HB'$ and $HA$ are the segments for which a mean proportional is to be found. By using the construction already fully explained, and shown for this case in Fig. 20, point $D'$ is found, which is projected parallel to $AC$ upon line $BC$, giving point $D$ in the plane of rupture. From $D$ the triangle of pressure is found as before.

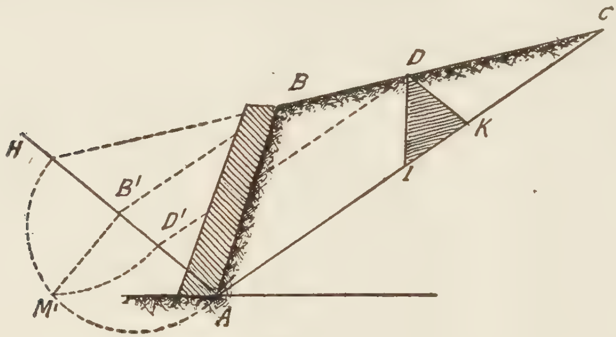



Fig. 20.

The last two constructions, (*b*) and (*c*), are very convenient when the angle of surcharge is large, as that point $C$ falls outside the sheet of paper.

**18.** Case 2. **No Surcharge.** — Professor Weyrauch has demonstrated analytically that when the upper surface of the embankment is horizontal, *i.e.* when there is no surcharge and the back of the wall is vertical, the pressure of the earth against the wall is normal to the back of the wall; in other words $\phi' = O$.

In this case the plane of rupture and the triangle of pressure can be determined very simply as indicated in Fig. 21. The directrix $AH$ now makes the angle $\phi$ with the back of the wall; then the two triangles $AHB$ and $AHC$

will be similar because they have the angle at $H$ in common, and the angle $HCA = BAH$ because both are equal to $\phi$; the third angle in each triangle is a right angle.

The triangles being similar,

$$HB : HA = HA : HC,$$

or

$$\overline{HA}^2 = HB \times HC.$$

But we know also (from Fig. 17) that

$$\overline{HD}^2 = HB \times HC.$$

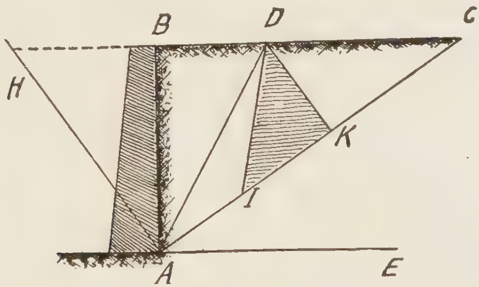



FIG. 21.

It follows from these two equations that $HA = HD$ and consequently the angles $HAD$ and $HDA$ are equal. Now, since $HC$ and $AE$ are parallel, angle $HDA$ equals angle $DAE$ as they are alternate interior angles; hence we may write the equations:

$$\angle HAD = \angle DAE,$$

or,

$$\angle HAB + \angle BAD = \angle DAC + \angle CAE,$$

or, what is same,

$$\phi + BAD = DAC + \phi,$$

whence, finally,

$$\angle BAD = \angle DAC.$$

That is to say, the plane of rupture $AD$ bisects the angle between the back of the wall and the natural slope of the earth.

When point $D$ has been found, the procedure for drawing the triangle of pressure is the same as already described. From $D$ draw $DK$ parallel to the directrix $AH$ to intersect the plane of repose at $K$. From $K$ lay off on $KA$ a segment $KI = KD$, and draw $ID$. Then triangle $DKI$ is the triangle of pressure.

**19.** Case 3. **Embankment with Maximum Surcharge; Angle of Surcharge equal to Angle of Repose.** — When the angle of surcharge is equal to the angle of repose, *i.e.* the top of the embankment is parallel to the natural slope of the soil, the lines $BC$ and $AM$ (Fig. 22) are parallel, which means that point $C$, their intersection, is at infinity. Under these conditions the plane of rupture will coincide with the plane of natural slope $AM$.

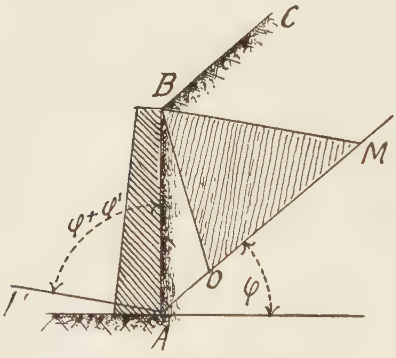


Fig. 22.

The triangle of pressure $DIK$, being contained between the two parallels $BC$ and $AC$, can be constructed at any point. Draw the directrix $AH$; from $B$ draw $BM$ parallel to $AH$; make $MO = MB$ and unite $O$ with $B$. The triangle $BMO$ is the triangle of pressure.

**20.** Case 4. **Embankment with Irregular Surcharge (Top of Embankment a Polygonal Profile).** — Suppose the earth be limited above by two planes whose traces $BG$ and $GC$,

Fig. 23, form a broken line. It is necessary to change the broken profile into a straight line by the method of transformation of figures into simpler equivalent ones as given by geometry. This reduces the problem to one of those already discussed (Cases 1 to 3).

In our Fig. 23 produce $GC$ to the left, connect $A$ with $G$, and from $B$ draw $BF$ parallel to $AG$. The triangle $AGF$ is equivalent to the triangle $ABG$ and can be taken in its stead. The problem is thus reduced to the problem

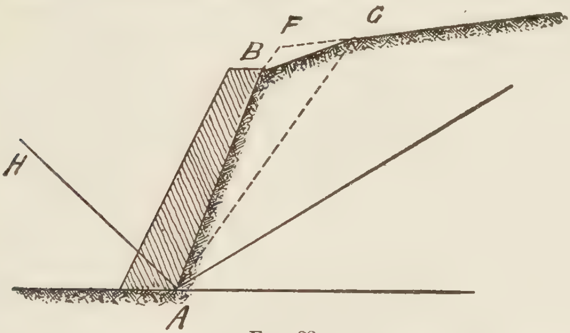



FIG. 23.

of Case 1, and is solved as follows: from $F$ draw $FM$ parallel to $AH$, giving point $M$ on the surface of repose $AC$. With $MC$ as diameter draw a semicircle, lay a tangent to it from $A$, $AT$, locate point $K$ on $AC$ by making $AK = AT$, and from $K$ draw $KD$ parallel to $AH$, giving point $D$ on the upper surface $FC$. Lay off the segment $KI = KD$ on $AC$, and connect $I$ with $D$, which gives the triangle of pressure $KDI$.

When the upper line of the embankment, instead of being a broken line as simple as the one just considered, consists of many segments, it still may, in every case, be reduced to an equivalent straight line by the same methods as indicated above.

**21.** CASE 5. **Embankment with Irregular Surcharge (Top of Embankment of Curvilinear Profile).** — When the embankment is bounded above by a cylindrical surface whose trace in the plane of section is a curve, the problem cannot be solved directly, but may be solved by trial.

According to Professor Rebhann's demonstration the problem can be reduced to a relatively simple one, viz. to find on the curved profile $BC$ a point $M$ such that the following equation be satisfied:

$$\text{area } ABM = \text{area } AMK,$$

the line $MK$ being drawn parallel to $H$. Along the curve $BC$, Fig. 24, take a large number of successive points I, II,

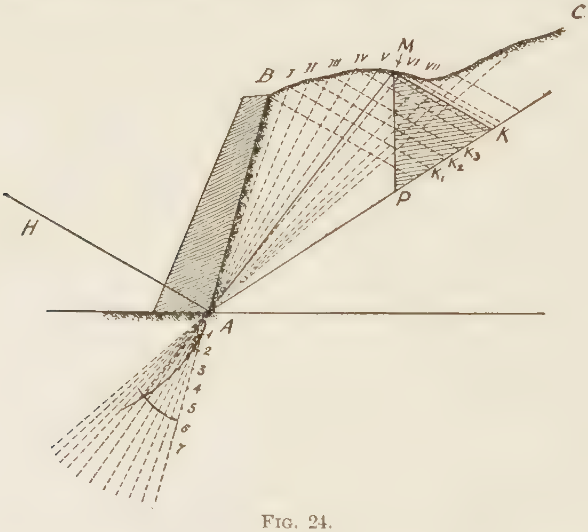



FIG. 24.

III, ··· etc., so close together that the sectors $AB$I, I$A$II, II$A$III, ··· etc., can be considered as triangles and their areas easily calculated. Assume, then, any convenient scale of reduction $b$ for these areas such that on dividing each area by

$b$ we obtain a number and may therefore represent the areas by segments of straight line. On $AB$, or in our case on $AB$ produced, lay off the segments $A$, 1; 1, 2; 2, 3, proportional to the areas of the sectors $BA$ I, I $A$ II, II $A$ III, etc., so that the line $A_3$, for instance, when multiplied by $b$ will represent the area between $AB$ and $A$ III.

Now with center at $A$ lay off the segment $A_1$ on the line $A$ I produced, $A_2$ on line $A$ II produced, $A_3$ on the line $A$ III produced, etc. Draw a curve through the points so found. The radius vector of this curve represents to scale the area of the prism of earth between the wall $AB$ and the radius vector itself.

Also, from $A$ draw the directrix $AH$ making the angle $\phi + \phi'$ with the back of the wall. From each one of the points I, II, III, . . . into which the curvilinear profile has been divided, draw lines I $K_1$, II $K_2$, III $K_3$, etc., parallel to the directrix $AH$; the triangles I $AK_1$, II $AK_2$, III $AK_3$, etc., are thus obtained. Measure the area of each one of these triangles, reduce these areas on the same scale $b$, and lay off the resulting values on the corresponding radii downward from $A$. Join the points so obtained by a smooth curve. Now each radius vector divides the embankment into two parts, a sector and a triangle; for example, the radius $A$ III gives the sector $BA$ III and the triangle III $AK_3$. These different areas corresponding to each radius are represented by the two curves. Where the curves intersect, it is evident the area of the sector is equal to the area of the triangle. But according to the theory of Professor Rebhann, this is the condition which defines the plane of rupture. Consequently the point $M$ (corresponding to point $D$ of the preceding cases) is found by uniting the point of intersection of the two curves of

area with $A$, and producing it till it strikes the curvilinear profile of the top of the embankment at $M$.

The triangle of pressure is found as before. From $M$ draw $MK$ parallel to the directrix $AH$, lay off on the plane of natural slope a segment $KP = MK$, join $M$ and $P$, and the resulting triangle $MPK$ is the triangle of pressure. The area of this triangle, multiplied by the weight of a cubic foot of the material, gives the thrust of the earth against the retaining wall, per lineal foot of wall.

Of course this method is slightly inaccurate, since the sectors $BA$ I, I $A$ II, etc., were considered as triangles, that is, the short arcs, $B$ I, I II, etc., were taken as straight lines. But by making the segments $B$ I, I II, etc., sufficiently short, the error may be reduced to any desired degree.

## Variation of Pressure with Height of Wall; Intensity of Pressure; Center of Pressure

**22.** In the following section we will investigate the manner of variation of the pressure against a retaining wall.

Suppose the point $A$, the foot of the wall, is moved upward along $AB$, *i.e.* the wall is decreased in height. For the new height of wall, both the plane of sliding and the triangle of pressure will be altered to determine these quantities under the new conditions, the construction previously used is to be repeated. Since all the angles remain the same, and the respective lines remain parallel to their original positions, the resultant figures will be similar. Therefore, also, the base as well as the altitude of the triangle of pressure will be proportional to the height of the wall $AB$, and in consequence the value of the pressure $P$ must be proportional to the square of the heights of wall.

In Fig. 25, for example, if $A'$ is the middle point of $AB$, from $A'$ draw $A'C'$ and $A'D'$ parallel to $AC$ and $AD$ respectively; the triangles $ABC$ and $A'B'C'$ are similar, also the triangles $A'BD'$ and $ABD$, having their sides respectively parallel. But in similar triangles the homologous sides are proportionals, and since $A'B = \frac{1}{2} AB$ by construction, the other sides of the triangles are in the same proportion, *i.e.* $BD' = \frac{1}{2} BD$ and $BC' = \frac{1}{2} BC$. In the similar triangles $A'D'C'$ and $ADC$, the altitudes $D'M'$ and $DM$ are in the same proportion as the sides, that is, $D'M' = \frac{1}{2} DM$. Then, the isosceles triangles $D'K'I'$ and $DKI$, having two sides respectively parallel and the included angle equal, are similar, and since $D'K' = \frac{1}{2} DK$, all the other corresponding sides will be in the same proportion, or $K'I' = \frac{1}{2} KI$. Now, the area of the triangle of pressure $K'D'I'$ corresponding to the height of a wall $A'B = \frac{1}{2} AB$ is given by the formula, area $= \frac{1}{2} D'M' \times K'I'$. Since $D'M' = \frac{1}{2} DM$ and $K'I' = \frac{1}{2} KI$, we have, area $D'K'I' = \frac{1}{2} \frac{DM}{2} \times \frac{KI}{2} = \frac{1}{2} \frac{DM \times KI}{4}$.

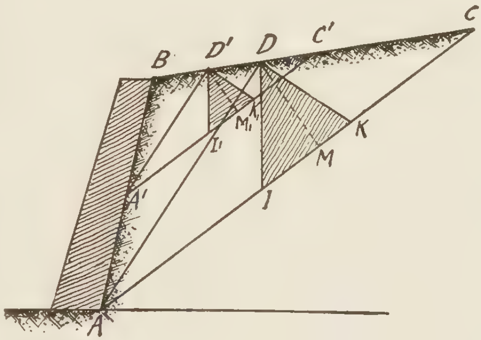



FIG. 25.

In other words, the area of the triangle $D'K'I'$ is one fourth the area of the triangle $DKI$. But the area of the triangle $DKI$ multiplied by $\gamma$, the unit weight of the material, gives the total pressure against the wall $AB$, while the corresponding pressure for a wall of the height $A'B = \frac{1}{2} AB$ will be one

fourth the area of the same triangle multiplied by $\gamma$, or the pressure is one fourth as great. Similarly, for a wall one third the original height, the pressure is one ninth, for a wall of double height the pressure is four times as great, etc. Thus we see that the pressure of earth against a retaining wall is proportional to the square of the height of the wall.

**23. Intensity of Pressure.** — We may also derive the principle that the intensity of pressure upon any element of the back of the wall (represented by the line $AB$) is proportional to the distance of the element below the top of the wall.

Let $y$ be the distance of any point on the back of the wall $AB$ below the top $B$. The total pressure above point $y$ being proportional to the square of the height $y$ can be represented by $cy^2$, where $c$ is a constant coefficient. Now, if $y$ is increased by an infinitesimally small quantity $dy$, the total pressure increases correspondingly by the quantity $2\,cydy$; making $dy = 1$, the total pressure increases by the amount $2\,cy$, which is directly proportional to the height $y$. Consequently the pressure per unit of height between $y$ and $y + dy$ is given by $2\,cy$, which is proportional to the height $y$.

The intensity of pressure at the middle point of the back of the wall is obtained by dividing the total pressure against the wall by the height of the wall. For if half the height of the wall be called $y$, the total pressure above the middle point is $P = cy^2$, and the intensity of pressure at the middle point, as just found, is $p_1 = 2\,cy$, or

$$p_1 = 2\,c\frac{AB}{2} = cAB. \qquad (1)$$

Also, the total pressure upon the back of the wall for its full height is $$P = cy'^2,$$
but in this case $$y^1 = AB,$$
so that $$P = c\overline{AB}^2.$$

The average pressure on $AB$ is obtained by dividing this equation by $AB$, which gives

$$p = \frac{P}{AB} = cAB.$$

Comparing this with equation (1) we see that $p = p'$, or the intensity of pressure at midheight is exactly equal to the average pressure against the entire wall $AB$.

The value of the intensity of pressure at the lowest point of the wall, at $A$, is double the intensity of pressure at the middle point $A'$. For, as above shown, the intensity of pressure at any point is expressed by the general formula $2\,cy$. At the point $A$, therefore, the intensity of pressure is $2\,cAB$, while at $A'$ (Fig. 25) it is $CAB$; the former is twice the latter. But also, since the intensity of pressure at midheight equals the mean intensity of pressure on the entire wall, as above shown, we see that the intensity of pressure at the base of the retaining wall is twice the average intensity of pressure against the wall.

The total pressure $P$ is given by the area of the triangle of pressure, $DKI$, multiplied by the weight of a cubic foot of the material, $\gamma$. If we call $a$ the altitude of the triangle, $IK$ being its base, the total pressure $P$ therefore is

$$P = \frac{1}{2}\gamma a \cdot IK.$$

Using this value of $P$ to express the average intensity of pressure $p$, and the intensity of pressure at the base $p'$, we have,

$$p = \frac{1}{2}\frac{\gamma a \cdot IK}{AB} \text{ and } p' = \frac{\gamma a \cdot IK}{AB}.$$

These values of the intensity of pressure at the middle and the base of the wall can be represented graphically. In Fig. 26, from $A$ draw a line perpendicular to $AB$ and on it lay off

a segment $AZ = IK$. Along $AB$ lay off another segment $AL = a$. Connect $L$ with $Z$; the triangle $ALZ$ is equivalent to the triangle of pressure $DKI$. Now connect $B$ with $Z$ and from $L$ draw $LV$ parallel to $BZ$.

The triangles $ABZ$ and $ALV$ are similar, consequently

$$\frac{AV}{AZ} = \frac{AL}{AB}.$$

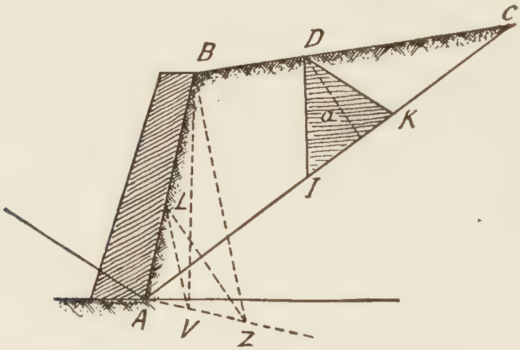



FIG. 26.

But by construction $AL = a$ and $AZ = IK$, so that

$$\frac{AV}{IK} = \frac{a}{AB},$$

from which we find that $\quad AV = \frac{a \times IK}{AB}.$

We found above that the intensity of pressure at the base is $p' = \frac{\gamma a \times IK}{AB}$; therefore $p' = \gamma AV$, or in other words the intensity of pressure at the base of the wall $AB$ is given by the segment $AV$ multiplied by $\gamma$, the specific weight of the soil.

The intensity of pressure at the middle point of $AB$ is $p = \frac{1}{2}p'$. This will be represented by one half of the segment $AV$ multiplied by the weight of a cubic foot of soil:

$$p = \frac{1}{2}p' = \frac{1}{2}\frac{Va \cdot IK}{AB} = \frac{1}{2}\gamma AV = \gamma\frac{AV}{2}.$$

**24. Center of Pressure.** — In the right triangle $ABV$ the width of the triangle at any point represents the intensity of pressure at that point; that is, ordinates drawn from $AB$ to the line $BV$ represent graphically the different values of $p = 2\,cy$. The total pressure upon the back of the wall will be given by the sum of the pressures on the successive unit areas, that is, it is equal to the area of the triangle $BVA$. It is applied at the center of gravity of the triangle $ABV$, at $\frac{2}{3}AB$ from the vertex $B$, and at $\frac{1}{3}AB$ from $A$. Hence we may say in general:

The point of application of the total pressure on the back of a retaining wall is at one third the height from the base.

The triangles $ABV$ and $DKI$ are equivalents, for $\triangle DKI = \triangle LAZ$ because constructed with equal bases and altitudes.

$$\triangle LAZ = \triangle LAV + \triangle LVZ$$

and as $$\triangle LVZ = \triangle LVB,$$

(having equal bases and altitudes, their vertices being on lines parallel), it follows that

$$\triangle LAZ = LAB + LVB = BAV.$$

Hence $$\triangle DKI = LAZ = BAV.$$

## The Earth Pressure represented by a Line

**25.** The pressure against a retaining wall can be represented by the length of a line drawn to a determined scale.

Let $h\gamma_m$ be the scale of reduction, in which $h$ may be taken as equal to 1, while $\gamma_m$ is the unit of weight of the material used in the construction of the retaining wall. Also, let

$p =$ the length of the line representing the pressure,

$a =$ the altitude of the triangle of pressure $DKI$,

$\gamma$ = weight of the unit of volume of earth in the embankment. Then we can write

$$p = \frac{P}{h\gamma_m} = \frac{\triangle DKI \times \gamma}{h\gamma_m} = \frac{IK \times a}{2h\frac{\gamma_m}{\gamma}}.$$

In Fig. 27, $\alpha$ is the angle that $DD'$ makes with $AH$ or its parallel $DK$; then

$$IK = DK = \frac{a}{\cos\alpha}.$$

Substituting this value of $IK$ in the former formula, we get

$$p = \frac{a^2}{2h\frac{\gamma_m}{\gamma}\cos\alpha}.$$

For simplicity call the denominator $2h\frac{\gamma_m}{\gamma}\cos\alpha = n$, then

$$p = \frac{a^2}{n}.$$

Now the angle $\alpha = \phi' - \delta$, equal to the angle that the direction of pressure makes with the horizontal. This is evident if from $A$ and $D$ we draw the vertical lines $AT'$ and $DT$; for the angles $KDT$ and $HAT'$ are equals; but $HAT' = \phi + \phi' - \delta$ and $TDK = TDD' + D'DK = \phi + D'DK$, since the angle $TDD' = \phi$ (because the sides are respectively perpendicular); hence the angle $D'DK = \alpha$ will be equal to $\phi' - \delta$.

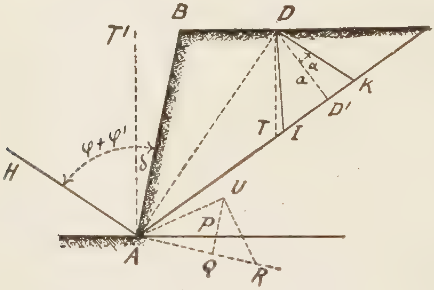


Fig. 27.

The following graphical construction can now be developed: on the horizontal line, Fig. 27, lay off a segment $AP = 2\,h\,\frac{\gamma_m}{\gamma}$. Then from $A$ draw a line $AQ$ making with $AP$ an angle $\alpha = \phi' - \delta$. From $P$ draw $PQ$ perpendicular to $AQ$. Then

$$AQ = AP \cos \alpha,$$

and substituting the value of $AP$,

$$AQ = 2\,h\,\frac{\gamma_m}{\gamma} \cos \alpha = n.$$

Along the perpendicular line $QP$ produced lay off a segment $QU = DD' = a$. Connect $U$ with $A$ and from the point $U$ draw the line $UR$ perpendicular to $AU$ meeting $AQ$ produced at the point $R$. In the triangle $AUR$ it is easily seen the perpendicular $UQ$ will be the mean proportional between the two segments $AQ$ and $QR$, that is,

$$\overline{UQ}^2 = AQ \times QR;$$

but $UQ = a$, and $AQ = n$, so that

$$QR = \frac{a^2}{n} = p.$$

Thus, with the scale of reduction selected as described, the length of the line $QR$ represents the total pressure against the wall $AB$.

It is observed that when the value of $n$ increases the pressure $P$ will decrease. Now, $n$ increases with a decrease of $\alpha = \phi' - \delta$. Since $\phi'$ is a constant, the variation of $P$ will depend exclusively upon the value $\delta$, which represents the inclination of the back of the wall from the vertical. In other words, the pressure will assume different values according to the various inclinations given to the back of the wall.

The pressure is smaller when the wall is inclined toward the slope of the embankment, and the pressure is least when $\phi' = \delta$.

When the inclination of the wall is outward, the angle $\delta$ changes sign, so that we will have $\alpha = \phi' + \delta$. The value of $n$ will then decrease with increasing values of $\delta$, and the value of the pressure $P$ will increase greatly.

From these considerations we may deduce that a wall is in better condition to resist the pressure when it is built with its back inclined toward the embankment. In practical work, walls are usually built with the back vertical, so that $\delta = O$. But walls with a slope of $\frac{1}{8}$ or $\frac{1}{10}$ will be more economical, quite as convenient, and will not be difficult to construct.

## Effect of Cohesion on Pressure against Retaining Walls

**26**. Hitherto, the problem of earth pressure against retaining walls has been considered without regard to the cohesive force of the soil or backing material. This was done for several reasons, chiefly because retaining walls are usually built to support filled earth, in which case the backing earth is practically devoid of cohesion. Even in the case of a wall supporting a bank of earth in its natural position, the cohesion is usually neglected, because the value of the cohesion is easily altered by atmospheric influences, while the force of friction remains almost constant in the same embankment. However, the forces of friction do not enter into action until the cohesive power of the material has been entirely destroyed or overcome.

It is a known fact that a bank of earth endowed with cohesion exerts a smaller pressure against a retaining wall

than a similar bank of earth without cohesion. Although for simplicity of calculation and for added safety the cohesive force of the material is always neglected, yet there are cases in which it will be useful to determine how far the cohesion of the earth affects the total pressure against the wall.

Let *ABC*, Fig. 28, be a bank of earth in which *AB* is the back of the retaining wall, *AC* is the plane of natural repose, and *AD* is the plane of rupture. The triangle *DKI* is

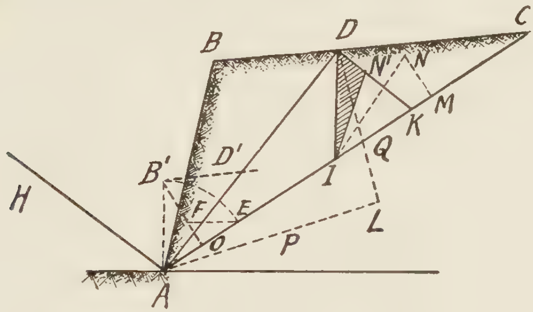



FIG. 28.

the triangle of pressure determined according to the methods indicated in the preceding sections.

The force of cohesion will prevent the prism of earth *ABD*, which causes the pressure, from sliding along the plane *AD*. Since the force of cohesion is uniformly distributed on this plane of sliding and is proportional to the area of the plane, it can be represented by the plane of sliding itself, and in our case will be represented by the line *AD*. In all these calculations the depth of embankment considered, in a direction perpendicular to the plane of the drawing, is equal to 1.

To determine the value of the cohesive force of the material it is necessary to know first the coefficient of cohesion.

For such a purpose cut a trench and find by experiment the vertical height up to which the material remains in equilibrium without support. Let $AB'$, in Fig. 28, be such a vertical height; then from $B'$ draw $B'O$ perpendicular to the line $AC$ of natural repose. Lay off a segment $AE = AB'$, and from $E$ draw the horizontal line $EF$. We know that the coefficient of cohesion is

$$K = \tfrac{1}{4} EF.$$

The force of cohesion per unit of surface is given by

$$C = K\gamma = \tfrac{1}{4}\gamma EF,$$

and the total force of cohesion acting to prevent the prism which causes the pressures from sliding along $AD$ will be given by

$$C = \tfrac{1}{4}\gamma EF \times AD.$$

Now resolve the force of cohesion, represented by the line $AD$, into its two components $Q$ and $P$, as was done for the prism $ABD$, p. 31. The direction of $Q$ was found to be at an angle $\phi$ with the normal to $AD$, hence it makes with $AD$ itself an angle $90° - \phi$. The direction of the pressure $P$ was found to make an angle $\phi'$ with the normal to the back of the wall, and consequently makes an angle $90° - \phi'$ with the back of the wall. Therefore draw the component $Q$ to make an angle $90° - \phi$ with $AD$ and the component $P$ to make an angle $90° - \phi'$ with $AB$.

Let $AL$ and $DL$ be the two components of the force of cohesion represented by $AD$.

The decrease of the earth pressure as a result of the cohesive force of the material will be represented by the component $AL$, that is, the pressure is decreased by a quantity

proportional to $AL$ and having the amount

$$C_1 = \frac{1}{4}\gamma EF \times AL,$$

which can be written

$$C_1 = \frac{1}{2}\gamma EF \times \frac{1}{2}AL.$$

Such a quantity can be graphically represented in the triangle of pressure as follows:

Along the line $AC$ from $I$ toward $IK$ lay off a segment $IM = \frac{1}{2}AL$. At $M$ erect a perpendicular to $AC$ and make $MN = EF$. Connect $N$ with $I$. Then the triangle $IMN$ represents the decrease in pressure due to the cohesion, because the area of this triangle multiplied by $\gamma$ will by construction be found equal to $C_1$.

Transform the triangle $IMN$ into an equivalent triangle having the base $IK$ common with the triangle of pressure, and the opposite vertex resting upon one of its sides. Let the triangle $IKN'$ be equivalent to $IMN$. The former pressure represented by the total area of the triangle $DKI \times \gamma$ is reduced through the effect of the cohesion by a quantity

$$C_1 = \triangle IKN'\gamma.$$

The remaining pressure against the wall, then, is represented by

$$P_1 = \triangle IDN' \times \gamma.$$

If instead of the total available value of the force of cohesion, only a portion of this force is relied upon, the amount of the pressure decrease will be determined in a similar manner, except that instead of the whole value of the cohesion only $\frac{1}{2}$ or $\frac{1}{3}$ should be taken, depending upon the factor of safety used. Thus, in the case just considered, if it were required to know the decrease of the pressure which would

result by the influence of half the available force of cohesion, the decrease of pressure is changed

from $$C_1 = \tfrac{1}{4}\gamma EF \times AL$$ to

$$C_{11} = \tfrac{1}{8}\gamma EF \times AL.$$

This quantity is represented graphically on the triangle of pressure by laying off from $I$ toward $K$ a segment equal to $\frac{1}{4} AL$, erecting a perpendicular and making $MN = EF$ the resulting triangle, which can be easily converted into another having $IK$ for base, and the vertex along $DK$ represents the decrease of pressure.

When the bank of earth stands in equilibrium on the slope $AB$ for the full height of the retaining wall to be constructed, the force of cohesion entirely counteracts the pressure. This means that the triangle $IMN$ is equal to the triangle of pressure $IDK$, and consequently that there is no pressure at all on the back of the wall. In this case the wall may be constructed of any thickness whatever, without regard to questions of earth pressure.

## The Pressure of Passive Resistance of the Earth

**27.** In the whole of the preceding discussion we have considered the pressure of the earth against a retaining wall on the assumption that the wall is stable and immovable, and that the pressure is caused by the tendency of the earth to move out of its original position. But it may happen that the wall exerts a pressure against the backing earth tending to force it back out of its position. In this case the earth exhibits an enormous resistance, far greater than the pressures heretofore considered. This phenomenon is observed occasionally in the abutments of bridges, where the arches

tend to push the abutments outward; if the thrust of the arches is greater than the active outward pressure of the earth, it forces the abutments back and develops the so-called passive pressure of the backing earth.

The conditions of the problem are the same as those already considered, except that they are reversed.

Let a wall *AB*, Fig. 29, exert a pressure against the backing earth greater than the active outward pressure of

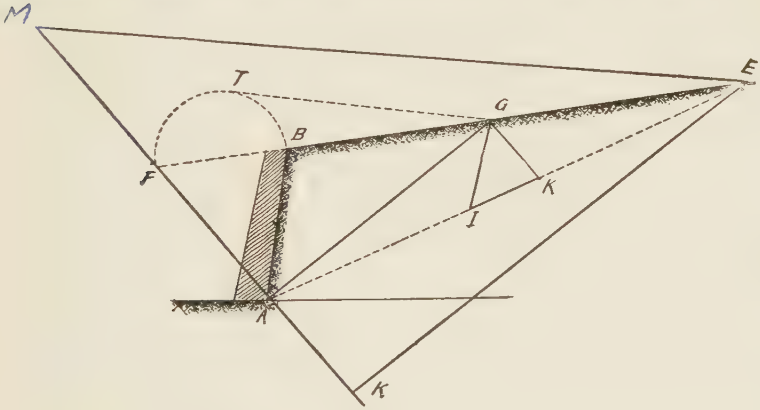



FIG. 29.

the earth. Then a prism *ABE* will tend to separate from the mass and slide upward along the plane of rupture. In this case we have the same force as heretofore, but the weight of the prism *ABE* will now oppose the ascending movement, and the resistances of friction along the surfaces *AB* and *AE* will now act downward; consequently the values of $\phi$ and $\phi'$ will change sign. In Fig. 29, *AB* being the back of the wall, *ABE* is the embankment as limited above by the surface *BE*, and *AF* is the trace of the plane of natural slope, passing through *A* but sloping downward to the right because $\phi$ is negative.

Draw the directrix $AG$ making with $AB$ an angle equal to $\phi + \phi'$ to the right, since $\phi$ and $\phi'$ have changed signs, whereas in the case of active pressure the directrix extended to the left of $AB$. Produce $BE$ until it intersects the plane of natural repose $AF$ at $F$; then on $BF$ as diameter describe a semicircle. From $G$ draw the line $GT$ tangent to this semicircle and make $GE = GT$. Connect $E$ with $A$; then the line $AE$ will represent the plane of sliding of the prism $ABE$.

Having located the plane of rupture it is easy to determine the value of the pressure. From $E$ draw the line $EK$ parallel to the direction $AG$, intersecting the line of natural repose $AF$ at $K$. From $K$ lay off along $KF$ a segment $KM = KE$ and connect $M$ with $E$. The triangle $KEM$ will be the triangle of passive pressure or the triangle whose area multiplied by the weight of a cubic foot of the material will give in pounds the value of the resistance of the bank of earth to the inward pressure of the wall.

The intensity of the passive pressure, or the pressure per square foot of wall, can be found in the same manner as for the active pressure, as follows:

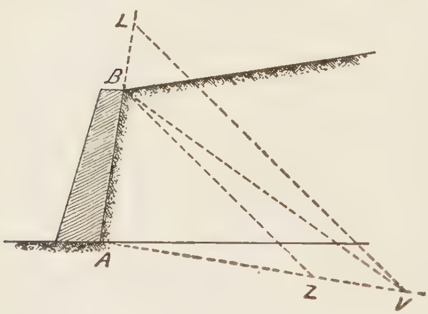


Fig. 30.

In Fig. 30, along the line $AZ$ drawn perpendicular to the back of the wall $AB$, lay off a segment $AZ = KM$, from $A$ along $AB$ produced lay off another segment $AL = a$, the altitude of the triangle $KEM$. Join $B$ with $Z$ and from $L$ draw $LV$ parallel to $BZ$. Then

$$AV : AZ = AL : AB,$$

or, substituting for $AZ$ and $AL$ their values,

$$AV : KM = a : AB,$$

that is,

$$AV \times AB = a \times KM,$$

from which

$$AV = \frac{a \times KM}{AB}.$$

Since the intensity of pressure at the base of the wall $p$ is equal to twice the average pressure, or twice the area of the triangle of pressure multiplied by the unit of weight of the material and divided by the total height of the wall $AB$, we find

$$p = \frac{Va \times KM}{AB} = \gamma AV.$$

Then the intensity of pressure at the middle of the height of the wall, which is equal to the average pressure along $AB$, is

$$p' = \frac{1}{2}p = \frac{1}{2}\,\frac{\gamma a \times KM}{AB} = \frac{1}{2}\gamma AV.$$

In the triangle $ABV$, the ordinates measured perpendicular to $AB$ represent the intensities of pressure at the various points of the wall. It follows that the center of pressure is at the center of gravity of the right triangle $ABV$, that is, at $\frac{1}{3}$ of $AB$ from $A$.

In order to compare the active and passive pressures, in Fig. 29, draw the triangle of active pressure $GIH$ in the usual manner. This is found to be much smaller than the triangle representing the passive pressure. The difference increases with increase of the angles $\phi$ and $\phi'$, and decreases

with decrease of the same angles; when both these angles are equal to zero, as in the case of liquids, then the active and passive pressures are equal.

The difference in magnitude of the active and passive pressures depends also upon the inclination and direction of the back of the wall $AB$. Thus, for instance, in walls inclined toward the outside, the difference will be found to be smaller than in walls either vertical or inclined toward the embankment.

In many actual cases the passive pressure of earth has been observed to be 3, 4, 10, even 20 times greater than the active pressure, and under the proper conditions it may be still larger.

# CHAPTER III

## RETAINING WALLS (continued): ANALYTICAL METHODS

### Rebhann's Method

**28.** Besides the demonstration of the graphical process for calculating earth pressure as given in the preceding chapter, Professor Rebhann has given also an analytical demonstration of his theory in the following manner:

Referring to Fig. 31, let $AB$ be the back or interior face of a retaining wall, $BC$ the limiting surface of the embankment, $AD$ any possible plane of rupture passing through $A$, and represent by $W$ the weight of the prism $ABD$. The prism $ABD$ in sliding along $AD$ will exert a pressure against the face $AB$ of the retaining wall and against the plane of sliding $AD$. Calling $P$ and $Q$ the reactions of these two planes, for equilibrium the three forces $W$, $P$, and $Q$, must meet in a point. Therefore, resolve the forces $W$, $P$, and $Q$ into their components parallel and perpendicular to $AD$ respectively. Then for equilibrium we must have: (1) the algebraic sum of the components parallel to $AD$, as well as the sum of those perpendicular to $AD$, must be equal to zero; and (2) the algebraic sum of the moments of the forces with respect to any point must be equal to zero. That is,

$$R_3 - R_2 - R = 0, \tag{1}$$

$$N_3 + N_2 - N = 0, \tag{2}$$

$$Ww + Pp - Qq = 0, \tag{3}$$

where $w$, $p$, and $q$ are the distances from $A$ of the forces $W$, $P$, and $Q$ respectively.

From (1) and (2) we have

$$R_3 - R_2 = R, \tag{4}$$

$$N_3 + N_2 = N, \tag{5}$$

but

$$R = N \tan \phi = (N_3 + N_2) \tan \phi, \tag{6}$$

so that

$$R_3 - R_2 = (N_3 + N_2) \tan \phi. \tag{7}$$

Evidently the one sliding plane, or in other words the actual plane of rupture, will be that for which the fraction

$$\frac{R_3 - R_2}{N_3 + N_2}$$

has the greatest value. If we denote by $\alpha$ the angle $DWB'$ between the plane $AD$ and the vertical passing through $W$,

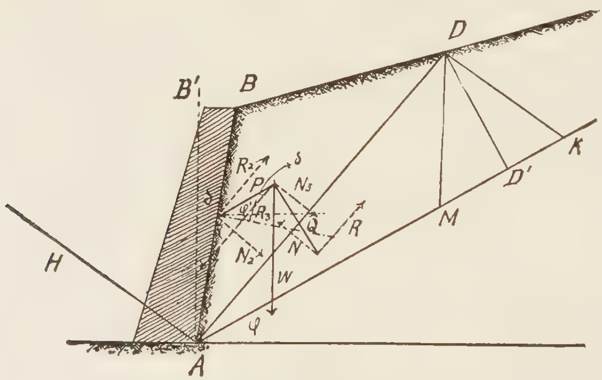



Fig. 31.

we can represent $R_3$, $R_2$, $N_3$, and $N_2$ in terms of $\alpha$; and if we then differentiate the above fraction in respect to $\alpha$, and put the differential equal to zero,

$$\frac{d}{d\alpha}\left\{\frac{R_3 - R_2}{N_3 + N_2}\right\} = 0, \tag{8}$$

we will obtain the condition giving the maximum value of the fraction, and therefore the condition which finds the plane of rupture.

To do this, we write

$$R_3 = W \cos \alpha,$$

$$R_2 = P \sin (\alpha + \phi' - \delta),$$

$$N_3 = W \sin \alpha,$$

$$N_2 = P \cos (\alpha + \phi' - \delta).$$

Substituting these values in (7),

$$W \cos \alpha - P \sin (\alpha + \phi' - \delta)$$
$$= [W \sin \alpha + P \cos (\alpha + \phi' - \delta)] \tan \phi,$$

from which we deduce

$$P = \frac{W \cos \alpha - W \sin \alpha \tan \phi}{\sin (\alpha + \phi' - \delta) + \cos (\alpha + \phi' - \delta) \tan \phi}$$

$$= W \frac{\cos (\phi + \alpha)}{\sin (\alpha + \phi' + \phi - \delta)}, \tag{9}$$

or

$$\frac{P}{W} = \frac{\cos (\phi + \alpha)}{\sin (\alpha + \phi + \phi' - \delta)}, \tag{10}$$

and

$$\frac{W}{P} = \frac{\sin (\alpha + \phi + \phi' - \delta)}{\cos (\phi + \alpha)}. \tag{11}$$

Substituting in (8) the values of $R_3$, $R_2$, $N_3$, $N_2$, we have

$$\frac{d}{d\alpha}\left[\frac{W \cos \alpha - P \sin (\alpha + \phi' - \delta)}{W \sin \alpha - P \cos (\alpha + \phi' - \delta)}\right] = 0.$$

Differentiating,

$$\left[\frac{dW}{d\alpha} \cos \alpha - W \sin \alpha - P \cos (\alpha + \phi' - \delta)\right]$$
$$\times [W \sin \alpha + P \cos (\alpha + \phi' - \delta)]$$

$$- [W \cos \alpha - P \sin (\alpha + \phi' - \delta)]$$
$$\times \left[\frac{dW}{d\alpha} \sin \alpha + W \cos \alpha - P \sin (\alpha + \phi' - \delta)\right] = 0,$$

which can be written

$$[W \sin \alpha \cos \alpha + P \cos (\alpha + \phi' - \delta) \cos \alpha - W \cos \alpha \sin \alpha +$$
$$P \sin (\alpha + \phi' - \delta) \times \sin \alpha] \frac{dW}{d\alpha} - W \sin \alpha + P \cos (\alpha + \phi' - \delta)^2$$
$$- P \sin (\alpha + \phi' - \delta)^2 = 0.$$

This can be reduced as follows:

$$P \cos (\phi' - \delta) \frac{dW}{d\alpha} - W^2 + 2\, WP \sin (\phi' - \delta) - P^2 = 0.$$

Dividing by $WP$ and changing signs,

$$\frac{W}{P} - 2 \sin (\phi' - \delta) + \frac{P}{W} - \frac{\cos (\phi' - \delta)\, d}{d\alpha} = 0. \qquad (13)$$

But $$dW = \tfrac{1}{2} \overline{AD}^2 d\alpha\gamma \text{ or } \frac{dW}{d\alpha} = \tfrac{1}{2} \overline{AD}^2 \gamma,$$

which, substituted in (13), gives

$$\frac{W}{P} - 2 \sin (\phi' - \delta) + \frac{P}{W} - \frac{\cos (\phi' - \delta)}{2\, W} \cdot \overline{AD}^2 \gamma = 0.$$

Now substituting for $\frac{W}{P}$ and $\frac{P}{W}$ their values as found in (10) and (11), we will have

$$\frac{\sin (\alpha + \phi + \phi' - \delta)}{\cos (\phi + \alpha)} - 2 \sin (\phi' - \delta) + \frac{\cos (\phi + \alpha)}{\sin (\alpha + \phi + \phi' - \delta)}$$
$$- \frac{\cos (\phi' - \delta)}{2A} \cdot \overline{AD}^2 \gamma = 0,$$

or

$$\frac{\sin (\phi + \alpha) \cos (\phi' - \delta) + \cos (\phi + \alpha) \sin (\phi' - \delta)}{\cos (\phi + \alpha)}$$

$$-\frac{2\sin(\phi'-\delta)\cos(\phi+\alpha)}{\cos(\phi+\alpha)}+\frac{\cos(\phi+\alpha)}{\sin(\phi+\alpha+\phi'-\delta)}$$

$$-\frac{\cos(\phi'-\delta)}{2W}\cdot\overline{AD}^2\gamma=0,$$

which reduces to

$$\frac{\sin(\phi+\alpha+\phi'-\delta)}{\cos(\phi+\alpha)}+\frac{\cos(\phi+\alpha)}{\sin(\phi+\alpha+\phi'-\delta)}$$

$$-\frac{\cos(\phi'-\delta)}{2W}\cdot\overline{AD}^2\gamma=0.$$

This can also be written as follows:

$$\frac{\sin(\phi+\alpha-\phi'+\delta)\sin(\phi+\alpha+\phi'-\delta)+\cos^2(\phi+\alpha)}{\cos(\phi+\alpha)\sin(\phi+\alpha+\phi'-\delta)}$$

$$=\cos(\phi'-\delta)\frac{\overline{AD}^2\gamma}{2W},$$

or

$$\frac{\sin^2(\phi+\alpha)\cos^2(\phi'-\delta)-\cos^2(\phi+\alpha)\sin^2(\phi'-\delta)+\cos^2(\phi+\alpha)}{\cos(\phi+\alpha)\sin(\phi+\alpha+\phi'-\delta)}$$

$$=\cos(\phi'-\delta)\frac{\overline{AD}^2\gamma}{2W},$$

which may be reduced to

$$\frac{\cos^2(\phi'-\delta)}{\cos(\phi+\alpha)\sin(\phi+\alpha+\phi'-\delta)}=\frac{\cos(\phi'-\delta)\overline{AD}^2\gamma}{2W},$$

and from this the value of $W$ is easily deduced as:

$$W=\frac{\cos(\phi+\alpha)\sin(\phi+\alpha+\phi'-\delta)}{\cos^2(\phi'-\delta)}\cdot\frac{\overline{AD}^2\gamma}{2}. \qquad (14)$$

Substituting this value in (9) the value of $P$, the pressure is found to be

$$P=\frac{\cos^2(\phi+\alpha)}{\cos(\phi'-\delta)}\times\frac{\overline{AD}^2\gamma}{2}.$$

The forces $P$, $W$, and $Q$ being in equilibrium, the sum of their horizontal components must be equal to zero, that is,

$$P \cos(\phi' - \delta) = Q \cos(\phi + \alpha),$$

or

$$Q = P\frac{\cos(\phi' - \delta)}{\cos(\phi + \alpha)} = \cos(\phi + \alpha)\frac{\overline{AD}^2\gamma}{2}. \tag{15}$$

In the triangle $ADK$, Fig. 31, from $D$ draw $DD'$ perpendicular to $AK$ and $DM$ vertically. Then the angle $ADD'$ will be equal to $(\phi + \alpha)$; for angle $ADM = B'AD$, their sides being respectively perpendicular, and $D'DM$ is equal to $\phi$ for the same reason, so that

$$D'DM = ADM + MDD' = \alpha + \phi.$$

Further, $DM$ is parallel to $AB'$, while $DK$ is parallel to $AH$; therefore $MDK = HAB_1$. But $HAB_1 = \phi + \phi' - \delta$; therefore angle $MDK = \phi + \phi' - \delta$. Since $MDK = MDD' + D'DK = \phi + D'DK$, it follows that $D'DK = \phi - \delta$.
Now in the triangle $AKD$ we have

$$DD' = AD\cos(\phi + \alpha) \tag{16}$$

and

$$AK = \frac{\sin(\phi + \alpha + \phi' - \delta)}{\cos(\phi' - \delta)}AD. \tag{17}$$

Consequently,

$$\triangle ADK \cdot \gamma = \frac{\cos(\phi + \alpha)\sin(\phi + \alpha + \phi' - \delta)}{\cos(\phi' - \delta)} \times \frac{\overline{AD}^2\gamma}{2} =$$

$$W = \triangle ABD\gamma, \tag{18}$$

which means that

*The plane of rupture $AD$ divides the surface $ABDK$ into two equivalent parts.*

Draw $AF$ perpendicular to $DK$; then

$$AF = AD\sin(\phi + \alpha + \phi' - \delta). \tag{19}$$

Dividing equation (16) by equation (19) we get

$$\frac{DD'}{AF}=\frac{\cos(\phi+\alpha)}{\sin(\phi+\alpha+\phi'-\delta)}=\frac{P}{W}.$$

Since by construction $DK=KM$,

$$\frac{\triangle KDM}{\triangle KDA}=\frac{DD'}{AF}=\frac{\cos(\phi+\alpha)}{\sin(\phi+\alpha+\phi'-\delta)}=\frac{P}{W};$$

but equation (18) shows that $\triangle KDA\gamma=W$. Therefore $KDM\gamma=P$, or, in other words:

The pressure of the earth against a retaining wall is equal to the weight of a prism having the triangle $KDI$ as a base, or

$$P=\tfrac{1}{2}\gamma KM\times DD'.$$

Finally, $$\triangle ADD''=\tfrac{1}{2}AD''\cdot DD',$$

and since by construction $AD''=AD$, while from equation (16) $DD'=AD\cos(\phi+\alpha)$, we have

$$\triangle ADD''=\tfrac{1}{2}\overline{AD}^2\cos(\phi+\alpha),$$

which is the value obtained for $Q$ in equation (15). Therefore

$$Q=\triangle ADD''\gamma,$$

in other words, the reaction $Q$ of the plane of rupture is equal to the weight of a prism of earth having the triangle $ADD''$ as base.

It will be apparent from the three equations $W=\gamma\cdot ABD=\gamma\cdot ADK$, $P=\gamma\cdot MDK$, and $Q=\gamma\cdot ADD''$, that

$$W:P:Q=AK:DK:AD.$$

We can therefore express the relations between these quantities by saying:

The weight of the sliding prism, the reaction of the retaining wall, and the reaction of the plane of rupture are in the same proportion as the sides of the triangle $AKD$.

## Formulas of Rankine and Weyrauch

**29.** Various authors have given different theories of earth pressure against retaining walls, based upon entirely different principles. Some authorities even discard all theories on the ground that they involve many assumptions and therefore cannot give reliable results. It is true that there has not yet been devised a method that will give in absolute and unexceptionable manner the value of the pressure against a retaining wall; but since the various approved methods lead to almost the same result, any one of them is capable of giving a result within certain limits satisfactory. In any actual case the engineer may, by comparing the assumptions made in the course of the calculation with the particular conditions of the bank of earth under consideration, secure a fair judgment of the pressure that the wall to be designed has to resist, and will be enabled at the same time to select a suitable factor of safety for the design.

After the pressure of the earth has been determined graphically according to the theory of Professor Rebhann as given in the preceding chapters, it will be instructive to calculate the pressure by another method and compare the results.

Professors Rankine and Weyrauch have studied the problem of earth pressure by analytical methods. Their theories require a long and complicated discussion, and for lack of space they are therefore not given here. The resulting formulas only are given.

**Rankine's Formulas.** — Professor Rankine assumes that the inclination of the pressure is always parallel to the upper line of the profile of the embankment, or, in other words, that it is parallel to the surcharge, and he considers the back of the wall vertical. If we denote by

$W$, weight of the earth per cubic foot,

$h$, height of the embankment in feet,

$\alpha$, the angle of surcharge,

and $\phi$, the angle of repose of the earth, he gives, for the general case,

$$P = \frac{Wh^2}{2} \times \cos\alpha \frac{\cos\alpha - \sqrt{\cos^2\alpha - \cos^2\phi}}{\cos\alpha + \sqrt{\cos^2\alpha - \cos^2\phi}}.$$

For level-top embankments, *i.e.* where $\alpha = 0$, this reduces to

$$P = \frac{Wh^2}{2} \times \frac{1 - \sin\phi}{1 + \sin\phi}.$$

For surface sloping at the angle of repose, *i.e.* when $\alpha = \phi$, it becomes

$$P = \frac{Wh^2}{2}\cos\phi.$$

**Weyrauch's Formulas.** — Professor Weyrauch has dealt with all possible cases of retaining walls, not only those of different surcharges, but also those of the inclined wall, the wall leaning either forward or backward. Using, in general, the same symbols as heretofore, *i.e.*

$P =$ total pressure of the earth against the wall.

$\phi =$ angle of repose of the material.

$\phi' =$ angle of direction of pressure with the normal to the back of the wall.

$\alpha =$ angle of the surcharge with the horizontal.

$\delta$ = angle of the back of the wall with the vertical taken either positive or negative, according as the slope of the wall will fall either to the left or right of the vertical.

$h$ = height of the wall.

$w$ = weight of earth per unit of volume.

$\beta$ = angle between plane of rupture and the vertical, and, referring to Fig. 32, he gives

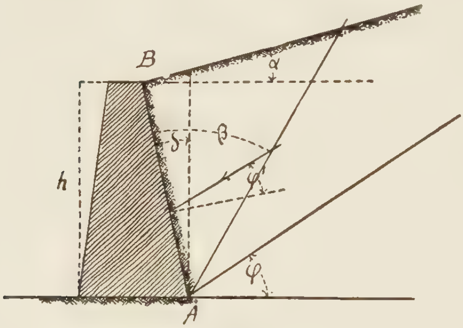



FIG. 32.

**General Formulas.**—*Embankment with any surcharge.*

Wall leaning forward, *i.e.* away from the embankment,

$$P = \left(\frac{\cos(\phi - \delta)}{(1 + n)\cos\delta}\right)^2 \times \frac{wh}{2\cos(\phi' - \delta)},$$

in which

$$n = \sqrt{\frac{\sin(\phi + \phi')\sin(\phi - \alpha)}{\cos(\phi' + \delta)\cos(\delta - \alpha)}},$$

and the value of $\phi'$ is deduced from the equation

$$\tan\phi' = \frac{\sin(2\,\delta - \alpha) - A\sin 2\,(\delta - \alpha)}{A - \cos(2\,\delta - \alpha) + A\cos 2\,(\delta - \alpha)},$$

in which the value of $A$ is given by

$$A = \frac{\cos\alpha - \sqrt{\cos^2\alpha - \cos^2\phi}}{\cos^2\phi}$$

Back of the wall vertical, $\delta = 0$,

$$P = \left(\frac{\cos\phi}{1 + n}\right)^2 \times \frac{wh^2}{2\cos\phi'}.$$

Back of the wall inclined toward the embankment, $\delta$ negative,

$$P = \left(\frac{\cos(\phi + \delta)}{(1+n)\cos - \delta}\right)^2 \times \frac{wh^2}{2\cos(\phi' + \delta)},$$

the values of $n$, $\phi'$, and $A$ being deduced from the formulas given above, except that $\delta$ is negative.

*Embankment with no surcharge, i.e.* $\alpha = 0$,

$$P = \frac{\tan\delta}{\sin(\delta + \phi')} \times \frac{wh^2}{2},$$

in which the value of $\phi'$ is given by

$$\tan\phi' = \frac{\sin\phi \sin 2\,\delta}{1 - \sin\phi\cos 2\,\delta}.$$

When the back of the wall is vertical, $\delta = 0$, and therefore by the formula last given, $\phi'$ will also equal zero and the value of the pressure reduces to

$$P = \tan^2\left(45^\circ - \frac{\phi}{2}\right)\frac{wh^2}{2}.$$

If the wall is required to retain water instead of earth, so that $\phi$ is equal to zero, then the above formula reduces to

$$P = \frac{wh^2}{2},$$

which is the well-known formula for the pressure of water against a dam.

The pressure cannot be calculated when the wall is inclined inward.

*Embankment with surcharge parallel to the natural slope;* $\alpha = \phi$.

$$P = \left(\frac{\cos(\phi - \delta)}{\cos\delta}\right)^2 \times \frac{wh^2}{2\cos(\phi' + \delta)},$$

in which $\phi'$ is obtained from

$$\tan \phi' = \frac{\sin \phi \cos (\phi - 2\,\delta)}{1 - \sin \phi \sin (\phi - 2\,\delta)}.$$

When the back of the wall is vertical, $\delta = 0$, and then $\phi = \phi'$, the value of the pressure is simplified to the form,

$$P = \tfrac{1}{2} \cos \phi\, wh^2.$$

When the back of the wall is inclined inward toward the embankment, the value of $\delta$ becomes negative and the general formula will be

$$P = \left(\frac{\cos (\phi + \delta)}{\cos - \delta}\right)^2 \times \frac{wh^2}{2 \cos (\phi' - \delta)},$$

the value of $\phi'$ being deduced from

$$\tan \phi' = \frac{\sin \phi \cos (\phi + 2\,\delta)}{1 - \sin \phi \sin (\phi + 2\,\delta)}.$$

**30.** The different values of the earth pressure determined by graphical computation according to the theory of Professor Rebhann are very close to those obtained from the formulas of Rankine and Weyrauch. This may be seen from the following tables prepared from the class work by Mr. W. J. Miller.

In the following tables are given in pounds the value of the earth pressure against walls of different heights. The walls were considered to be vertical, leaning 5° forward and inclined 5° toward the embankment. In the first table the bank of earth was considered with no surcharge, or when $\alpha = 0$; in the second table when the surcharge made with the horizontal an angle of 10°, or $\alpha = 10°$. Finally, in the last table, the surcharge was considered parallel to the line of natural slope of the material, or when $\alpha = \phi$. In every case the weight of the earth was assumed to be 96 lb. per cubic foot, and the angle of natural slope $\phi = 30°$.

## TABLE I

| Height of Wall | Vertical Method, or $\delta = 0°$ | | | Wall Inclined Outward, or $\delta = 5°$ | | Wall Inclined Inward, or $\delta = -5°$ |
|---|---|---|---|---|---|---|
| | Graphical Method | Analytical Method | | | | |
| Ft. | Rebhann | Weyrauch | Rankine | Rebhann | Weyrauch | Rebhann |
| 25 | 9747 | 9990 | 10000 | 12420 | 12392 | 8748 |
| 30 | 14035 | 14400 | 14400 | 15552.08 | 15148 | 11282.88 |
| 35 | 19392 | 19533.56 | 19600 | 21168 | — | 17053 |
| 40 | 24953 | 25497.6 | 25674.4 | 27648 | — | 22394.8 |
| 45 | 31570.2 | — | — | 35992 | — | 28441 |
| 50 | 38988 | — | — | 43200 | — | 35992 |
| 55 | 49152 | 48399.5 | 48400 | 53868 | 50050.5 | 47990 |
| 60 | 56142 | 60048.6 | 60276.5 | 62238 | 61834 | 50388.4 |
| 65 | 65899 | — | 67606 | 73008 | — | 58968 |
| 70 | 76416.4 | 78306.9 | 78400 | 84672 | 81089.9 | 72610.56 |
| 75 | 87723 | — | — | 97200 | — | 78732 |
| 80 | 99809 | — | — | 110592 | — | 93726 |
| 85 | 112675 | 114332 | 114576 | 124848 | 122987 | 101126.88 |

## TABLE II

| Height of Wall | Vertical Wall, or $\delta = 0°$ | | | Wall Inclined Outward, or $\delta = 5°$ | | Wall Inclined Inward, or $\delta = -5°$ | |
|---|---|---|---|---|---|---|---|
| | Graphical Meth. | Analytical Method | | | | | |
| Ft. | Rebhann | Weyrauch | Rankine | Rebhann | Weyrauch | Rebhann | Weyrauch |
| 25 | 9948 | 10370 | 10490 | 12309.5 | 12150 | 9117.12 | 9094 |
| 30 | 14470.8 | 14320.6 | 15094.8 | 18338 | 18527.5 | 12558 | 11648.6 |
| 35 | 20260.8 | 20168.4 | 20768 | 23461.8 | 22684 | 17472 | 18546.2 |
| 40 | 25712 | 26343 | 27287 | 29506 | 28287 | 23325 | 23804.8 |
| 45 | 33916 | — | — | 39512 | — | 29328 | — |
| 50 | 40176 | — | — | 47880 | — | 34884 | — |
| 55 | 49960 | 50304 | 51935.6 | 56544 | 54630 | 42408 | 43508 |
| 60 | 59197.4 | 59027.6 | 58947 | 66842.9 | 67834 | 50592 | 48915 |
| 65 | 67897.7 | 67606.4 | 68884.4 | 89310.2 | 86966 | 61036.8 | — |
| 70 | 80828.6 | 80661 | 82459.3 | 93225.6 | 92052.8 | 68372.7 | — |
| 75 | 90396 | — | — | 107730 | — | 78489 | — |
| 80 | 107623 | — | — | 122572 | — | 89303 | — |
| 85 | 121169.9 | 120866 | 120205 | 138954.8 | 138402.9 | 105327.4 | 105202 |

TABLE III

| | Vertical Wall, or $\delta = 0$ | | | Wall Inclined Outward, or $\delta = 5°$ | | Wall Inclined Inward, or $\delta = -5°$ | |
|---|---|---|---|---|---|---|---|
| Height of Wall | Graphical Method | Analytical Method | | | | | |
| Ft. | Rebhann | Weyrauch | Rankine | Rebhann | Weyrauch | Rebhann | Weyrauch |
| 25 | 26700 | 25942.5 | 25980 | 30290.8 | 31020 | 22140 | 23008.24 |
| 30 | 37315.2 | 37285.75 | 37368 | 43636.32 | 43632.2 | 31999.6 | 32111.33 |
| 35 | 50568.9 | 50922.56 | 50922.56 | 59904 | 60900 | 43394 | 43036 |
| 40 | 66048 | 66508.8 | 66508.8 | 77575.8 | 77731 | 47078.4 | 49788 |
| 45 | 83592 | 84175 | 84175 | 98277 | — | 71733.6 | — |
| 50 | 103200 | 103923.6 | 103923.6 | 121212 | — | 88560 | — |
| 55 | 124872 | 124389.2 | 125747.55 | 139200 | 138989 | 115055.2 | 116466.2 |
| 60 | 148761.6 | 149649.9 | 149644.8 | 174535 | 174570.88 | 127488 | 128835 |
| 65 | 174408 | 175668 | 175668 | 204848 | — | 148666.4 | — |
| 70 | 203616 | 203690.25 | 203690.25 | 237575.5 | 236320.6 | 174128.6 | 174731 |
| 75 | 232200 | — | 233828 | 272270 | — | 199250.8 | — |
| 80 | 264192 | — | 266044 | 252160.8 | — | 226718.6 | — |
| 85 | 298248 | 300339.2 | 300339.2 | 354394 | 354340 | 255938 | 257591 |

# CHAPTER IV

## THE DESIGN OF RETAINING WALLS

**31. Types of Retaining Walls.** — Retaining walls are built of a variety of sections. For convenience these may be grouped as follows:

I. Plain retaining walls.

II. Retaining walls with counterforts.

III. Retaining walls with buttresses.

Each group can be further subdivided into many varieties according to the inclination and shapes of the front and back of the wall.

**32. Plain Retaining Walls.** — The plain retaining wall may be constructed in various manners:

(*a*) With vertical front and back, as indicated in Fig. 33.

(*b*) With vertical front and inclined back, as indicated in Fig. 34.

(*c*) With vertical back and inclined front, Fig. 35.

(*d*) With both front and back inclined, but in opposite directions, as indicated in Fig. 36.

(*e*) With front and back inclined in the same direction, Fig. 37.

(*f*) With stepped back. In order to make the wall thicker toward the bottom, the back of the wall may be stepped while the front is plain, either vertical (Fig. 38) or inclined (Fig. 39). The back of the wall in the successive steps may be either vertical or inclined.

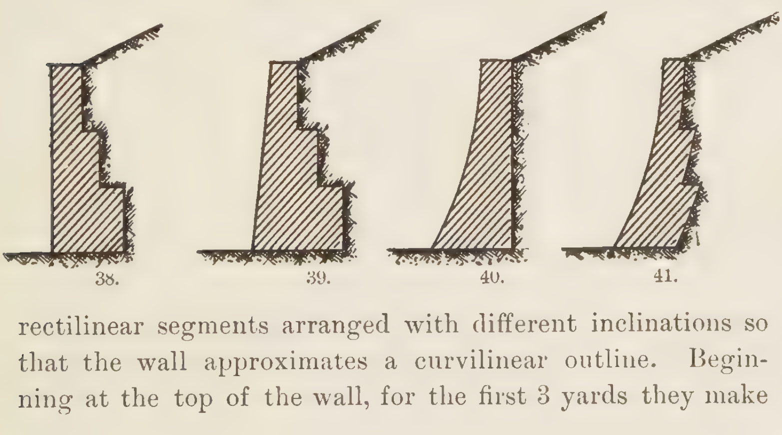
(*g*) With curved face. English engineers often make the back of the wall vertical, whether stepped or not, while the front of the wall is curved to a circular arc of radius gener-

33. 34. 35. 36. 37.

ally equal to twice the height of the wall, the center of the arc being located on the line of the top of the wall produced. Such a section is shown in Fig. 40.

(*h*) With curvilinear back and front. German engineers often make the front of the retaining wall circular, following the English manner, but at the same time make the back of the wall, which is usually of stepped construction, also circular and parallel to the front, as indicated in Fig. 41.

(*i*) French engineers have adopted a profile composed of



38. 39. 40. 41.

rectilinear segments arranged with different inclinations so that the wall approximates a curvilinear outline. Beginning at the top of the wall, for the first 3 yards they make

the inclination of $\frac{1}{5}$, for the second 3 yards, $\frac{1}{4}$, and for the residual height, no matter how high the wall, the constant inclination of $\frac{3}{10}$ of the height. The back of the wall is formed with steps about 1 ft. wide for every 5 ft. in height. The French profile is shown in Fig. 42.

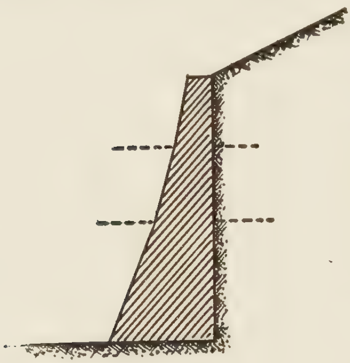


FIG. 42.

**33. Retaining Walls with Counterforts.** — Retaining walls with counterforts are similar to plain retaining walls as described above, but along the back of the wall, at intervals, are constructed blocks of masonry of larger dimensions reënforcing the wall proper. These blocks of masonry may be constructed on either rectangular or trapezoidal ground plan, as indicated in Figs. 42 and 43, and they may be vertical or inclined in profile. They may even have a stepped outline. Such counterforts are located usually from 12 to 20 ft. apart, center to center.

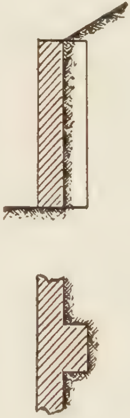


FIG. 43.

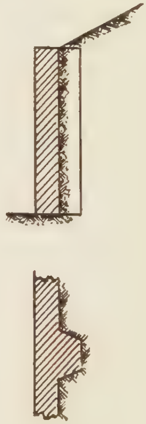


FIG. 44.

**34. Retaining Walls with Buttresses.** — Retaining walls with buttresses (Figs. 45 and 46) are very similar to those with counterforts, the only difference being that the pillars or projecting blocks of masonry are on the front of the wall instead of on the back. The buttresses may be rectangular

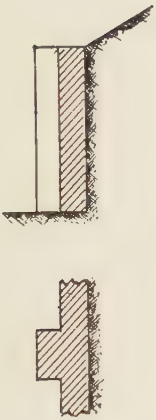


Fig. 45.

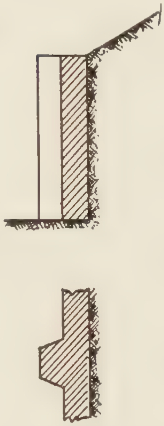


Fig. 46.

or trapezoidal in plan, and either vertical or inclined in profile, according to the shape of the front of the wall, but they are never recessed. Buttresses are spaced about the same as counterforts, viz., from 12 to 20 ft., center to center.

In very heavy soils to relieve the pressure of the earth against the retaining wall, horizontal arches sprung between the counterforts may be used. In such cases the counterforts must be designed strong enough to resist the pressure which is transmitted to them by the arches. The arches may be built in one row or in several rows, and the lowest may sometimes be inverted, as in Fig. 47.

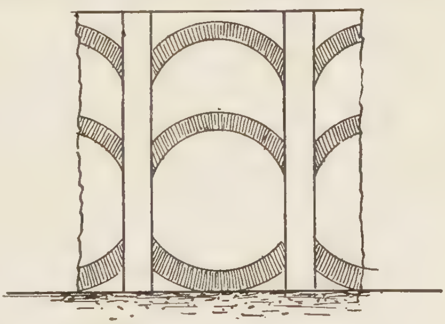


Fig. 47.

## The Equilibrium of Retaining Walls

**35.** The forces acting on a retaining wall are the pressure of the earth $P$ and the weight of the wall $W$. It has been demonstrated that the point of application of the earth pressure is at a distance above the base of one third the height of the wall. The weight of the wall is assumed to be concentrated at its center of gravity.

The pressure $P$ (Fig. 48) being inclined, making an angle $\phi'$ with the normal to the back of the wall, may be resolved into its vertical and horizontal components. The horizontal component thrusts against the wall, tending to produce both sliding and rotation; the vertical component, on the other hand, assists the weight $W$ in pressing down the different strata of the wall and thereby counteracting the overturning and sliding tendency of the horizontal thrust. Under the action of these forces the wall tends to overturn in whole or in part. The most dangerous section is usually the section at the base of the walls, since the earth pressure increases as the square of the height and therefore increases faster than the resisting forces, the weight of the wall. The tendency of the resultant $R$ of the forces $P$ and $W$ to approach the exterior of the wall is greatest at the base. For this reason the stability of the wall is always determined with respect to the section at the base, and the thickness of any wall to resist a given pressure is always calculated for the base of the wall.

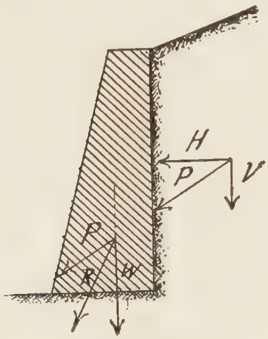


FIG. 48.

As a rule, a retaining wall is in less danger of failure by sliding than failure by overturning; consequently when a wall is safe against overturning it is generally safe against sliding.

In determining the thickness of retaining walls it is generally assumed that the wall is laid up dry, *i.e.* without mortar, resisting only by virtue of its own weight, and having no tensile strength. Tensile forces are produced in

a wall when the pressure is so near one edge of the wall and is so great that it tends to force down the material at this edge and raise up or tear apart the material at the other edge.

In order that a wall may be strong enough to resist the action of the forces $P$ and $W$ in all the probable sections of rupture, the following conditions must be satisfied:

1. The resultant $R$ of the forces $P$ and $W$ must fall within the middle third of the width of the base. This condition, while it makes the wall safe against overturning, also makes tensile stresses impossible.

2. The direction of the resultant $R$ must make a smaller angle with the normal to the section than the angle of friction of the masonry of the wall. This condition insures stability against failure by outward sliding.

3. The greatest compressive stress must be smaller than the crushing strength of the material composing the wall. This condition insures stability against crushing.

The stability of a retaining wall may be calculated in two different ways: either fixing the condition of equilibrium of the wall and adding a factor of safety, or requiring as a condition of safety that the resultant $R$ shall pass through the middle third of the base.

For calculations according to the first method, let $m$ = the lever arm of $P$, then $Pm$ will be the product of $P$ by its distance from the foot of the wall $A$, $w$ = the lever arm of $W$, and $Ww$ the product of $W$ by its distance from $A$.

The wall will be in equilibrium when

$$Pm = Ww.$$

The wall is safe if it is able to resist greater pressure than $P$. It is usual for this purpose to multiply the moment of

*P* by a factor of safety, which is generally fixed somewhere between 1.50 and 2.

Such a method, although giving good results, especially when a factor of safety of 2 is used, does not convey any idea of the distribution of stresses in the masonry. Thus, for example, a wall may be safe against overturning, being made to fulfill the condition $Pm = Ww$ with a factor of safety of 2 and yet it may crack and be ruined because the compressive stresses in the wall are concentrated on a small portion instead of being evenly distributed, and because tensile stresses are thus set up.

The second method is the more rational one; and when the engineer has given such proportions to the wall that the compressive stress remains below the crushing strength of the material, he may depend upon the stability of his wall.

## Determination of Width of Base by Graphical Methods

**36.** In the previous chapters, in calculating the pressure of the earth against a retaining wall, it has uniformly been assumed that the length of wall considered was one unit, consequently that the wedge which tends to slide down and thus causes the thrust against the back of the wall had a length of one unit. For this reason we have always used the area of the triangle of the base as equivalent to the volume of the wedge itself. Continuing this practice, we will deal with a retaining wall having a length of one unit, perpendicular to the plane of the cross-section (the plane of the paper). Then all calculations can be referred to the profile of the wall, since the area of the profile is numerically equal to the volume of a wall one unit long.

From what has gone before, the pressure of the earth against the back of the wall is fully known as to direction, magnitude, and position. When the weights of both the earth and the masonry per unit of volume are known, it becomes a very simple matter to oppose to the prism of earth causing the pressure a suitable resisting section of masonry.

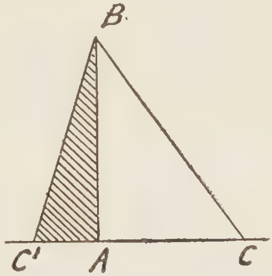


Fig. 49.

Let $ABC$ (Fig. 49) represent the cross-section of the active prism of earth, and denote by $\gamma_m$ and $\gamma_t$ the weight of a unit of volume of masonry and earth respectively. Then for equal weight we will have,

$$AC'\gamma_m = AC\gamma_t,$$

from which

$$AC' = \frac{AC\gamma^t}{\gamma_m},$$

then $ABC'$ will represent a mass of masonry of weight equal to the pressure of the earth.

A retaining wall will be in equilibrium when the moment of the weight is equal to the moment of the pressure. To design a safe wall it is necessary to multiply the moment of pressure by a certain coefficient $m$, called factor of safety; then we will have

$$Ww = Ppm,$$

the coefficient $m$ is usually made equal to 1.5, 2, or even 3.

**37.** Example 1. **Retaining Wall with Vertical Front and Back.** — Let $AB$ (Fig. 50) represent the back of the wall. From $A$ and $B$ draw the lines $AX$, $BX'$ horizontally; the

thickness of the wall is supposed to have been fixed by successive trials. Along the line $AX$ lay off a segment $A1$ equal to the unit of length, though it can be taken of any length, and erect a perpendicular, giving the first part of cross-section of the wall. Draw the force polygon, making the ray $oO$ equal twice the earth pressure, or $oO = 2P$ and parallel to the direction of the pressure itself, *i.e.* making an angle $\phi'$ with the normal to the back of the wall. In the present case, we will assume, it is proposed to build a wall capable of resisting twice the pressure of the earth, that is, a wall having a factor of safety of 2. For this reason we made $oO = 2P$. From $o$ draw a vertical line and lay off a segment $o1$ equal to the weight of the first part of the wall. Join 1 with $O$; this will be the resultant of the total pressure combined with the weight of this part of the wall. Then at one third of $AB$ draw the line of the earth pressure, produce it until it strikes at $I$ the center line of this section of the wall, and draw from $I$ a line parallel to the resultant $1O$.

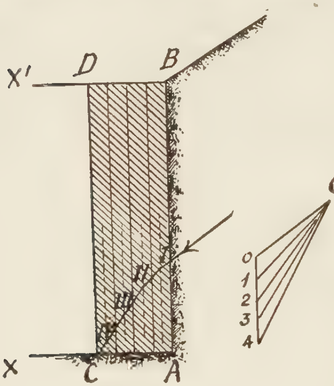


Fig. 50.

Again, along $AX$ take a second segment, and erect a perpendicular. Find the center of this second section of wall and produce the first resultant until it meets this center line at $II$. In the force polygon lay off a segment 1–2 equal to the weight of this second prism of wall. Draw the resultant $O2$, and from $II$ draw a parallel to this resultant. Proceed in the same manner by adding continu-

ously small sections of wall, until a point is reached where the last resultant strikes the base line $AX$ at $C$. Erect the perpendicular $CD$, the wall $ABCD$ will be the required wall, being in equilibrium under twice the actual earth pressure.

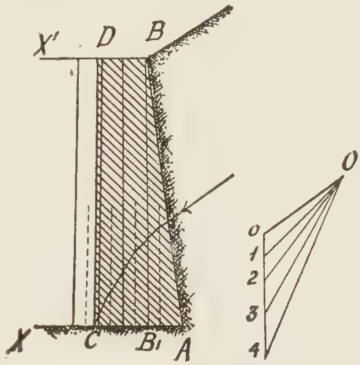


Fig. 51.

**38.** Example 2. **Retaining Wall with Vertical Front and Inclined Back.** — Let $AB$ be the back of the wall (Fig. 51). Draw the horizontals $AX$ and $BX'$. From the point $B$ drop the perpendicular $BB'$ and divide the undetermined rectangular portion to the left of $BB'$ into a number of small vertical stripes as above. Draw the force polygon to such a scale that the segment $o1$ will represent the weight of the triangular prism $BB'A$ and the segments 1.2, 2.3, 3.4, etc., the corresponding weights of the successive rectangular prisms of masonry. The construction will be perfectly similar to the one given in the case above except that the earth-pressure line is continued to the vertical through the center of gravity of triangle $ABB'$. Where the polygon of the resultant pressure meets the line $AX$ draw the perpendicular $CD$, the figure $CDAB$ will be the required section of the wall.

**39.** Example 3. **Retaining Wall with Inclined Front and Vertical Back.** — In Fig. 52, $AB$ represents the back of the wall and $CD$ the given inclination of the front. Draw the usual horizontal lines, and from $B$ draw $B1$ parallel

to $CD$. Along the upper line mark off the points $F$, 2, etc., and from these draw lines parallel to $CD$, the intervals will represent so many prismoids of masonry. Then draw the force polygon, and along the line of weights lay off the first segment, $O1$, to represent the weight of the triangular prism $AB1$, and then in succession the weights of the prismoidal strips lying to the left of $B1$. Find the center of gravity of the various prisms of masonry and draw the resultants in the usual manner, continuing them in each case to intersect the verticals through the centers of gravity. At the point $E$, where the polygon of resultants strikes the base line, draw the line $EF$ parallel to the given inclination $CD$. The figure $ABEF$ will be the cross-section of the required wall with the face parallel to the given inclination.

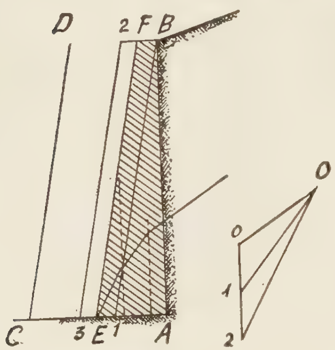


FIG. 52.

**40.** EXAMPLE 4. **Retaining Wall with Faces inclined in Opposite Directions.** — The line $AB$ (Fig. 53) represents the back of the wall. From $B$ draw a perpendicular and treat the first triangle as in Example 2. Then deal with the rest of the wall, up to the front face $CD$, according to the method followed in Example 3. The resulting profile $ABCD$ is the cross-section of the wall.

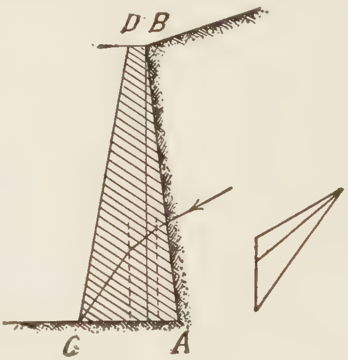


FIG. 53.

**41.** EXAMPLE 5. **Retaining Wall with Parallel Inclined Faces.** — The construction (Fig. 54) is perfectly similar to that of Example 1, being careful to use as vertical center lines the verticals through the centers of gravity of the several sections of the wall. When both faces are inclined in the same direction but are not parallel, the general method of Example 4 is to be used.

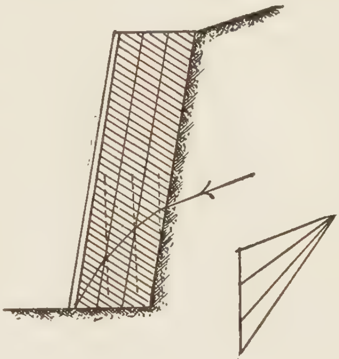


FIG. 54.

**42. Interpolation Method.** — All the above problems can be solved in another way, employing the principle that the required section of the wall, which is in equilibrium with the pressure (1.5, 2, or 3 times the actual pressure, according to the factor of safety), is included between two cross-sections of which one has excessive stability and the other deficient stability. In other words, we first determine a section of wall in which the resultant between the pressure and weight falls outside the base of the section; it is evident that such a wall will not be strong enough to resist the assumed pressure. Then we determine a section of wall in which the resultant falls within the base; such a wall will be too strong. The required cross-section will be between these two.

In Fig. 55, $ABCD$ and $ABEF$ are the two cross-sections thus determined, and $R$ and $R_1$ their respective resultants. From $C$ drop a perpendicular and lay off a downward segment $Cm = RC$. At $E$ erect a perpendicular and lay off upward a segment $En = ER'$. Connect $n$ with $m$; at the point of intersection with the ground line draw a line paral-

lel to *CD*. The figure *ABGH* will be the required section of the wall. In this construction it is assumed that as the line representing the front of the wall moves from *CD* to *EF*, the curves traced by points located in the manner *m* and *n* were located in a straight line. Really it is a curve of small curvature, but the error in assuming it to be straight is so small as to be negligible in practical work.

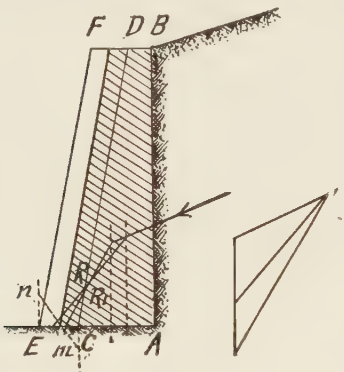



Fig. 55.

It may be required that the resultant of the actual earth pressure (not multiplied by a factor of safety) and the height of the wall should fall within the middle third of the base. In this case proceed in the following manner: after the thickness of the wall has been determined by any of the methods already explained, in the force polygon (Fig. 56), draw *OS* equal to the actual earth pressure and weight of the wall. On the back of the wall *AB* produce the line of pressure to intersect at *Q* the vertical dropped from the center of gravity of the figure *ABGH*. From *Q* draw a line parallel to the resultant *SM*, intersecting the base *AG* at *N*. Now if this resultant falls either within or just at the extreme end of the middle third of the base, the wall

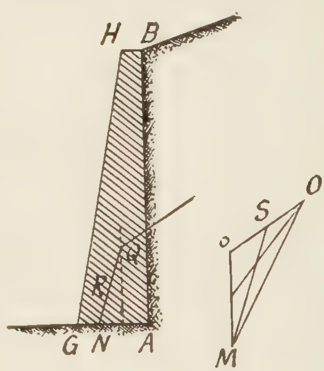



Fig. 56.

satisfies the given condition. But if it falls outside the middle third, the thickness of the wall must be increased until the new resultant does fall inside the middle third.

43. Example 6. **Retaining Walls with Counterforts.** — The graphical determination of retaining walls with counterforts and buttresses becomes very simple when such walls are considered as composed of two parts having different specific weights. The stability of the wall will then be determined per unit of length just as in the cases already reviewed. For the sake of simplicity in the graphical construction it is assumed that both counterforts and buttresses are located 12 ft. apart, center to center, and that they are 3 ft. wide.

Under retaining walls with counterforts, two cases may be considered: (1) the thickness of the wall being given, it is required to determine the thickness of the counterfort; or (2), the thickness of the counterforts being given, the thickness of the wall is to be determined. We will begin with the first case, and we will consider the condition when both wall and counterforts are vertical.

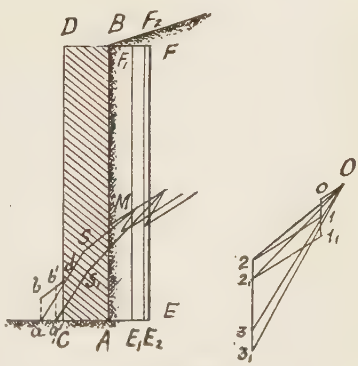



Fig. 57.

Referring to Fig. 57, $AB$ is the given thickness of the wall. It is required to determine the thickness $AE$ of the counterfort. This is done by successive trials, adding small slices of wall, just as in the examples already discussed. The pressure $P$ divides in the proportion of $\frac{3}{4}$ on the wall and $\frac{1}{4}$ on the

counterfort. Draw the force polygon in the usual manner. The pressure on the counterfort will be represented by $oO$; the weights of the various slices of the counterfort should be taken at $\frac{1}{4}$ the weight of equal sections of wall. Suppose $AE$ is the first size of counterfort we try. Its weight will be represented by $o1$ in the force polygon; draw the line 1.2 parallel to the pressure and equal to $\frac{3}{4}P$ (or $\frac{3}{4}2P$ in case 2 is used as factor of safety). Draw a perpendicular and lay off a segment 2.3 equal to the weight of the given wall; the line 3.0 will be the resultant. In the figure draw the line of pressure at $\frac{1}{3}E_1F_1$, produce it to meet the vertical from the center of gravity of the prism of masonry $ABE_1F_1$, and from this point draw a line parallel to the resultant $O3$, until meeting at $M$ the line of pressure which strikes the back of the wall $AB$ at $\frac{1}{3}$ the height from $A$. Draw from $M$ a parallel to the direction of pressure $O2$ of the force polygon, meeting at $S$ the vertical dropped from the center of gravity of the wall $ABCD$, and from $S$ draw a parallel to the resultant $O3$, meeting the ground line at the point $a_1$. Take a second segment $E_1F_1E_2F_2$, add its weight $1.1_1$ in the force polygon and proceed as above; we then find that the last resultant drawn from $S_1$ meets the ground line at the point $a_2$. At $a_1$ and $a_2$ erect vertical lines and lay off segments $a_1b_1$ and $a_2b_2$ equal to $AE_1$ and $AE_2$, respectively; connect $b_1$ and $b_2$ and produce this line until it intersects the line of the front of the wall at $d$. The length of the line $Cd$ represents the required thickness of the counterfort.

The prism of masonry of the counterfort $ABEF$ displaces a similar prism of earth which was previously calculated as exerting thrust against the wall. The actual

pressure against the counterforted wall will therefore be a little smaller than the value used.

(*b*) When the thickness of the counterfort *ABCD* is given and it is required to determine the thickness of the wall, the problem is easily solved. Draw the force polygon in Fig. 58 by laying off in the direction of the pressure a segment *mo* equal to $\frac{1}{4}P$ (or $\frac{1}{4}2P$ in case a factor of safety 2 is used); then draw a vertical line and lay off a segment *mn* equal to $\frac{1}{4}$ the weight of the counterfort, and from *n* in the direction of the pressure draw *nr* equal $\frac{3}{4}P$ (or $\frac{3}{4}2P$). From the point *r* draw a vertical line and lay off segments equal to the weights of the various slices of masonry *ABEF*, *EFGH*, etc., and proceed as in Example 1.

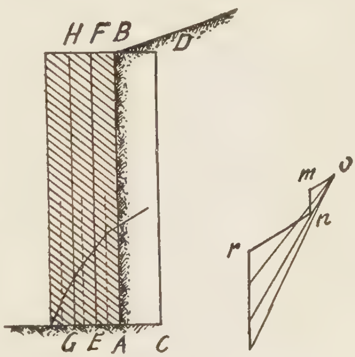



FIG. 58.

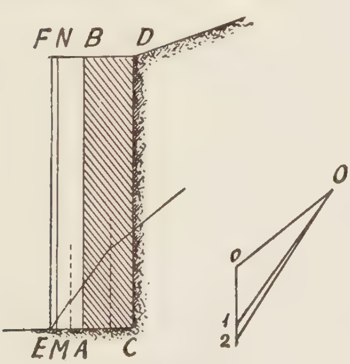



FIG. 59.

**44.** EXAMPLE 7. **Retaining Wall with Buttresses.** — The thickness of the wall *ABCD* (Fig. 59) being given, that of the buttress *ABEF* is easily determined as follows: draw the force polygon in the usual way, making *oO* equal to twice the actual earth pressure; then lay off vertically a segment *o*1 equal to the weight of the wall *ABCD*. Try a buttress of thickness *AM* and lay off in the

force polygon a segment 1–2 equal to one fourth the weight of the prism *AMNB*; draw the resultant 2.0. In the figure draw the pressure polygon and if the resultant strikes outside the base, try a new thickness of buttress. When the final resultant meets the ground line at the edge of or just within the base of the buttress the required thickness *AE* of the buttress *ABEF* will have been found.

45. EXAMPLE 8. **Retaining Wall with Inclined Buttresses.** — In case of inclined buttresses, the necessary thickness at the base is determined by trial. When the wall *ABCD* (Fig. 60) is given, the force polygon $oO1$ can be easily drawn; the center of pressure or resultant will certainly fall outside the base of the wall, otherwise there is no need of buttresses. Assume, then, any thickness *EC* for the buttresses and draw from *E* a line parallel to the given inclination *IK*; lay off in the force polygon a segment 1, 2 equal to one fourth the weight of the prism *CDEF*. From the point where the resultant emerging from the wall meets a vertical dropped from the center of gravity of *CDEF*, draw a line parallel to $o2$ of the force polygon. This will fall either within or without the wall; in the former case there is an excess of stability, and the thickness of the buttresses can be taken as *EC* or reduced by a new trial, while in the latter case it will be necessary to increase the thickness of the base, using the methods already indicated for walls.

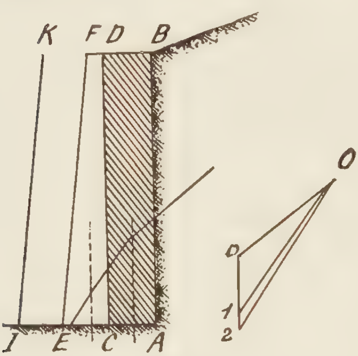


FIG. 60.

46. EXAMPLE 9. **Retaining Walls with Relieving Arches between Counterforts.** — In this case there will be back of the wall, besides the counterforts, the masonry arches and the earth packed between the arches. In order to determine the thickness of the counterforts and arches, it is necessary to determine the average specific weight of the structures and materials back of the wall.

Let $A$ = the area included between the center lines of two consecutive counterforts,

$A_1$ = the area occupied by the earth between the arches,

$A_1$ = the weight of the earth per cubic foot,

$A_2$ = the area occupied by masonry arches and counterforts, and

$A_2$ = the weight of the masonry per cubic foot.

The average specific weight back of the wall will be

$$W = \frac{A_1 w_1 + A_2 w_2}{A}.$$

After the unit of weight has been determined, the thickness either of the counterforts when the wall is given, or of the wall when the counterforts are given, is obtained as in the case of simple counterforts indicated above.

In the case of a retaining wall with relieving arches between the buttresses, the solution is similar to that indicated in Example 7 for determining the thickness of a buttress. In that example the buttresses taken as weighing only one fourth of the weight of the masonry of the wall, because they were only 1 yard wide and 4 yards apart. In the present case, however, the average specific weight of the buttresses and arches is to be calculated as

$$W = \frac{A_2 w_2}{A}.$$

When this value has been determined the process is quite the same as already explained, and the required thickness of buttress is found directly.

## Determination of Width of Base by Analytical Method

**47.** Suppose $ABCD$ (Fig. 61) to represent the cross-section of a retaining wall having inclined front and vertical back. The forces acting on the wall are:

(*a*) The outward pressure of the earth backing, $P$.

(*b*) The weight of the wall, $W$.

(*c*) The reaction of the foundation upward against the base $AC$ of the wall, which is equal and opposite to the resultant $R$ of the two forces, $P$ and $W$. This reaction will be denoted by $R'$.

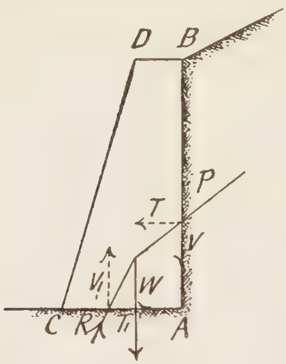


Fig. 61.

Resolve the forces $P$ and $R'$ into their horizontal and vertical components; call $T$ and $V$ the horizontal and vertical components of the pressure $P$, and $T'$ and $V'$ the horizontal and vertical components of the reaction $R'$.

When the wall is in equilibrium, the algebraic sum of the forces as well as the algebraic sum of their moments must be zero. Therefore

$$T - T' = 0,$$

$$W + V - V' = 0,$$

and

$$Tt + Ww - V'v = 0,$$

$t$, $w$, and $v$ being the lever arms of the forces $T$, $W$, and $V'$,

or the distances of their points of application from $A$, the interior edge of the wall.

From the above equations we have:

$$T = T',$$

$$W + V = V',$$

$$Tt + Ww = V'v.$$

The last equation gives directly the value of $v$,

$$v = \frac{Tt + Ww}{V'}.$$

Substituting to $V'$ its value $W + V$, we obtain

$$v = \frac{Tt + Ww}{W + V}. \tag{1}$$

The pressure $P$ makes an angle $\phi'$ with the normal to the back of the wall, and in the present case with the horizontal. Since $T$, the projection of $P$ on the horizontal, we have

$$T = P \cos \phi',$$

$$V = P \sin \phi'.$$

Substituting these values in equation (1), there results

$$v = \frac{Pt \cos \phi' + Ww}{W + P \sin \phi'}. \tag{2}$$

If from $A$ we draw a line perpendicular to the direction of $P$, its angle with $AB$ will be equal to $\phi'$; the length of this perpendicular will be $t \cos \phi'$, so that the latter term in equation (2) may be replaced by $m$. Then

$$v = \frac{Pm + Ww}{W + P \sin \phi'}. \tag{3}$$

In order that the resultant $R$ fall within the middle third of the base, $v$ must lie between $\frac{1}{3}$ and $\frac{2}{3}$ the width of the

base. Calling the base $t$, it will have the smallest allowable value when

$$v = \tfrac{2}{3}\,t,$$

or

$$t = \frac{3}{2}\,v = \frac{3}{2}\left(\frac{Pm + Ww}{W + P\sin\phi'}\right), \tag{4}$$

which is the general formula.

Assume the thickness of the wall at the top to be 0.6 of the width of the base (in practice it would rarely be made smaller). Then the weight $W$ of the trapezoidal section of the wall is expressed by the formula

$$W = \frac{t + 0.6\,t}{2}\,h\gamma.$$

Substituting this value in equation (4) and solving for $t$, the thickness of the wall at the base is determined directly, without resort to trial calculation.

The case considered is only a special case, however, since we assumed that the back of the wall is vertical. The general case is that in which both back and front of the wall are inclined. Let us analyze this case, assuming further that to increase the safety against sliding, the base of the wall is inclined at an angle $\alpha$ to the horizontal. The angle between the base and the back of the wall will be called $\beta$ (Fig. 62).

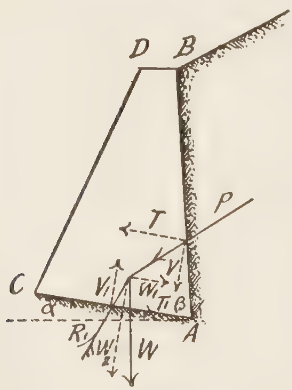


FIG. 62.

The external forces are the same as before; namely, $P$, $W$, and $R'$. Resolve each one into its components, normal and parallel to the base $AC$.

The forces being in equilibrium, we have

$$T - T_1 - W_1 = 0,$$
$$V + W_2 - V_1 = 0,$$
$$Pm + Ww - V_1 v = 0,$$

from which we deduce

$$T = T_1 + W_1,$$
$$V_1 = V + W_2,$$
$$V_1 v = Pm + Ww.$$

The last of these gives

$$v = \frac{Pm + Ww}{V_1}.$$

Substitute for $V_1$ its value $V + W_2$, we get

$$v = \frac{Pm + Ww}{V + W_2}.$$

Now the pressure $P$ (Fig. 63) makes an angle $\phi'$ with the normal to the back of the wall, and consequently it makes an angle $(90 - \phi')$ with the surface $AB$. Then $T$ makes with $P$ an angle $180 - \beta - (90° - \phi') = 90° - (\beta - \phi')$, so that the angle between $P$ and its vertical component $V$ is equal to $\beta - \phi'$. But $\beta = 90 - (\alpha + \delta)$; therefore

$$\beta - \phi' = 90 - (\alpha + \delta + \phi').$$

The vertical component $V$ of the pressure $P$ is therefore expressed by

$$V = P \cos(\beta - \phi)$$
$$= P \cos(90 - (\alpha + \delta + \phi')),$$

or

$$V = P \sin(\alpha + \delta + \phi').$$

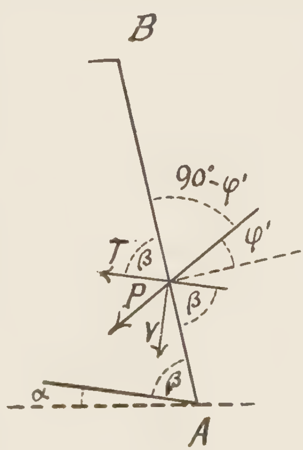



FIG. 63.

The angle between $W_2$ and $W$ is equal to angle $\alpha$, since the two angles have sides respectively perpendiculars; hence

$$W_2 = W \cos \alpha.$$

On substituting these values of $V$ and $W_2$ in equation 5, we find

$$v = \frac{Pm + Ww}{W \cos \alpha + P \sin (\alpha + \delta + \phi')}.$$

In order that the resultant shall pass through the outer edge of the middle third of the base, the base of the wall must be given a width of $t = \frac{3}{2} v$, or

$$t = \frac{3}{2} \frac{Pm + Ww}{W \cos \alpha + P \sin (\alpha + \delta + \phi')}.$$

On the other hand, if it is required that the wall shall be able to resist the pressure $P$ with a factor of safety of $n$ against overturning, we substitute $Pn$ for $P$ and make the base of the wall just wide enough so that the resultant strikes the outer edge of the wall; that is, we make $t = v$. In this case the formula becomes,

$$t = \frac{nPm + Ww}{W \cos \alpha + P \sin (\alpha + \delta + \phi')}.$$

# CHAPTER V

## DAMS

**48. Kinds of Dams.** — Walls intended to retain water instead of earth are called "dams." They can be considered as masonry structures erected for the purpose of raising the level of water.

Dams may be grouped in two classes: ordinary dams and submerged dams. Ordinary dams are those to retain or store water, forming a reservoir or lake. They are higher than the level of the water in the reservoir or lake. Submerged dams, on the other hand, have the object of raising the level of water in a river; since the water will continue to flow down the channel it rises sufficiently to overflow the structure, so that the dam will be entirely under water or submerged.

In regard to their manner of resisting the pressure of the water, dams may be classed as gravity dams and arched dams. Gravity dams are those which resist the pressure exclusively by their own weight. Arched dams, however, are those in which the pressure is transferred through the body of the masonry to the sides of the valley, against which the structure abuts. Arched dams are constructed in the shape of an arch having the two side hills for abutments, or skewback.

Another type of dam used in many instances is the reservoir embankment or earth embankment. These usually contain a masonry core wall, but as regards their static resistance against water pressure they are to be consid-

ered simple earth embankments, since the core wall is used only to prevent seepage of water through the embankment and to tie the mass of earth together, and not for the purpose of resisting the pressure of water. Earth embankments are not considered in this work.

Ordinary dams act exactly like retaining walls and by the use of the principles and methods already given it will be an easy matter to determine their proper proportions when we know the direction, magnitude, and point of application of the pressure which the structure is required to resist.

**49. Direction of the Water Pressure.** — The pressure of water is always perpendicular to the back of the dam. This fact was demonstrated by Pascal long ago. He performed an experiment in which he forced water into a hollow metal ball pierced with very small holes at various points. The water squirted out of all the holes with the same velocity, hence the same pressure. It was concluded that since the pressure in the water was equal in all directions, it must act in directions perpendicular to the surface inclosing it.

Referring to our discussion of earth pressure against retaining walls, it was shown that the direction of the pressure makes an angle $\phi'$ with the normal to the back of the wall, and we have always assumed that $\phi' = \phi$, the angle of natural slope of the material. When the material is water instead of earth, the angle of natural repose is equal to zero, so that $\phi'$ also will be equal to zero, and consequently the direction of the pressure will coincide with the normal to the back of the wall.

**50. Amount of the Pressure.** — In determining the pressure of earth against a retaining wall graphically it was found that the value of the pressure was given by the area

of a certain triangle multiplied by the unit of weight of the material. The triangle of pressure was shown to have one of its sides parallel to the directrix, a line drawn so as to make the angle $\phi + \phi'$ with the back of the wall, and another side lying in the surface of repose, these two sides being equal to each other. Since the construction used for this triangle of pressure is valid for all values of $\phi$ and $\phi'$, independently of the nature of the material, it will apply also to water, a material in which the angle of natural repose is equal to zero.

Suppose the line $AB$ (Fig. 64) represents the back of a dam and the line $DC$ the surface of water in the reservoir back of the dam. Since both $\phi$ and $\phi'$ are equal to zero, the surface of repose will coincide with the horizontal line $AE$, and the directrix coincides with the back of the dam $AB$. The triangle of pressure is then easily determined. The line $AD$ will be the side of the isosceles triangle parallel to the directrix, and by measuring a length equal to $AD$ on the horizontal line $AE$, the surface of repose, we determine the point $F$. Joining $D$ with $F$, there results the triangle of pressure $ADF$, whose area multiplied by the specific weight of water will give the total pressure against the back of the dam. Thus,

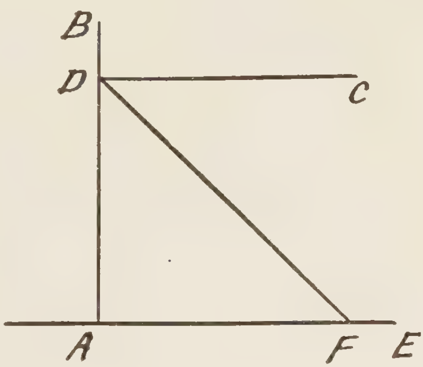


Fig. 64.

$$P = \triangle ADF \cdot \gamma,$$

which can be written also

$$P = \tfrac{1}{2}\, AD \times AF \times \gamma.$$

But $AD = AF = h$, the depth of water in the reservoir, substituting this value gives

$$P = \tfrac{1}{2}\, h \times h \times \gamma,$$

or

$$P = \tfrac{1}{2}\, h^2\gamma,$$

which is the well-known formula for determining the horizontal pressure of water against a dam.

**51. Point of Application of the Pressure.** — The point of application of the resultant pressure on the back of the dam is a distance above the base equal to one third the depth of water.

In Fig. 65, the triangle $ABC$ represents the total pressure of the water when the depth of water is equal to $AB$. For any other point, $D$, the total pressure on that part of the dam above the point will be given by the triangle $BDE$; calling $BD = h_1$, the total pressure represented by the triangle $BDE$ is expressed by $P = \tfrac{1}{2}\, h_1^2\gamma$. Thus in the case of water pressure as in the problem of earth pressure, the total pressure is proportional to the square of the height of the wall. In the triangle $ABC$ the ordinates measured parallel to $AC$ represent the intensities of pressure at the corresponding depths; thus the length of the line $AC$ represents the intensity of pressure at the depth $BA$, while the length of the line $DE$ represents the pressure per square foot at a depth $BD$. The total pressure on a small element of height near the

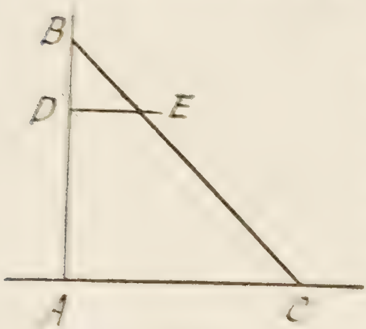


Fig. 65.

point $D$ is therefore measured by the area of the strip between the ordinates drawn at the upper and lower edges of the element, which is equal to the mean ordinate $DE$ multiplied by the height of the element. The triangle $ABC$ is the sum of all the pressure areas for all the successive elements of height composing the wall $AB$, or is equal to the total pressure. It follows that the resultant pressure acts through the center of gravity of triangle $ABC$, which is located one third of $AB$ above the base $A$. The location of this resultant is called the center of pressure.

The location of the center of pressure can also be determined analytically, as follows:

Represent any portion of the back of the dam by the area $ABCD$ (Fig. 66). The coördinates of points of this area will be denoted by $x$ and $y$, measured from $O$, in the water surface, $y$, representing the horizontal distance from the center line of the area, and $x$ the depth from the water surface down to the point in question. Consider a narrow vertical strip of width $y$. The center of pressure of this strip is at a depth $X$ below the surface, given by the following formula, for a total height of the strip equal to $h$:

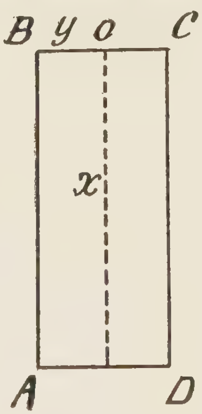


FIG. 66.

$$X = \frac{\int_0^h yx^2 dx}{\int_0^h yx dx}.$$

Since $y$ is constant, we may integrate for the rectangular area of the strip, thus,

$$\int_0^h x^2 dx = \left(\frac{x^3}{3}\right)_0^h = \frac{h^3}{3},$$

and

$$\int_0^h xdx = \left(\frac{x^2}{2}\right)_0^h = \frac{h^2}{2},$$

so that we have

$$X = \frac{y\int_0^h x^2dx}{y\int_0^h xdx} = \frac{y\frac{h^3}{3}}{y\frac{h^2}{2}} = \frac{2}{3}\frac{h^3}{h^2} = \frac{2}{3}h,$$

or the center of pressure is at $\frac{2}{3}$ the total height of the strip from the surface of water, or at $\frac{1}{3}$ the height from the base. The whole area $ABCD$ is composed of a large number of such strips side by side, as the center of pressure for each is at the same height, the resultant center of pressure is also at $\frac{1}{3}h$ above the base.

Dams fail from precisely the same causes as retaining walls. Like a retaining wall, a dam may fail in any one of three different ways, viz., by rotation or overturning, by sliding, and by crushing. To insure the stability of a dam, three conditions must therefore be satisfied:

1. The resultant of water pressure and weight must fall within the middle third of the base. This will give a factor of safety of 2 against overturning, or rotation around the exterior point of the base.

2. The angle made by the resultant with the perpendicular to the base must be smaller than the angle of friction between the foundation and the dam. This condition will insure stability against sliding.

3. The maximum intensity of pressure at the base of the dam must be less than the crushing strength of the material. This condition will insure safety against crushing. A suitable factor of safety must be employed here.

The same conditions must be satisfied at every horizontal section of the dam, since each section $BDE$ (Fig. 65) may be considered as an independent dam resting on the part $ADEC$ as a foundation.

## Theoretical Profile for Dams

**52.** In studying dams we will always deal with a slice one foot in length. Then the area of the cross-section is numerically equal to the volume of the portion considered, and the area of the triangle of pressure multiplied by the weight of a cubic foot of water is equal to the total pressure.

**53. Triangular Profile.** — The most economical profile for a dam, theoretically, is a triangular one, with downstream slope as given by the formula, demonstrated below,

$$\frac{t}{h} = \sqrt{\frac{\gamma_a}{\gamma_m}},$$

where $\gamma_a$ and $\gamma_m$ represent the weights of the unit of volume of water and masonry respectively, $t$ the thickness of the dam at the base and $h$ the total height of the structure (Fig. 67).

The pressure $P$ of water against the dam when the water surface is level with the crown is given by the formula,

$$P = \tfrac{1}{2}\,h^2\gamma_a.$$

Hydraulics teaches that the pressure of water on a submerged surface of any shape whatever equals the area of the surface multiplied by the vertical distance from its center of gravity to the surface of the water, multiplied by the weight

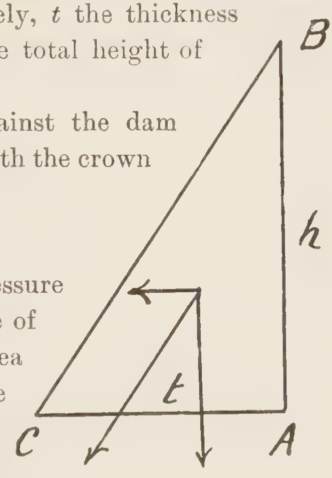



Fig. 67.

of a cubic foot of water. In the present case the water side of the dam being vertical, and the area of wetted surface being a rectangle whose width is one foot and whose height is equal to the depth of water, the center of gravity of the wetted area will be at the middle point of the height, or at $\frac{h}{2}$ below the water surface, while the area is $1 \times h = h$; the pressure $P$ will be, therefore,

$$P = \tfrac{1}{2} h \times h\gamma_a = \tfrac{1}{2} h^2\gamma_a.$$

The weight of the dam (Fig. 67) will be,

$$W = \tfrac{1}{2} th\gamma_m.$$

Dividing the pressure by the weight,

$$\frac{P}{W} = \frac{\frac{1}{2} h^2\gamma_a}{\frac{1}{2} th\gamma} = \frac{\gamma_a h}{\gamma_m t}.$$

The outer edge of the middle third of the base will be at $\frac{1}{3} t$ from $W$, since the vertical line through the center of gravity of a right triangle will meet the base at $\frac{1}{3}$ the width from the right angle. Then taking moments about the outer edge of the middle third of the base, the resultant moment must be equal to zero if the resultant pressure passes through the edge of the middle third as is demanded by the condition of stability against overturning; that is,

$$P \times \tfrac{1}{3} h - W \times \tfrac{1}{3} t = 0,$$

or

$$P \times \tfrac{1}{3} h = W \times \tfrac{1}{3} t,$$

from which we find the relation necessary for stability to be,

$$\frac{P}{W} = \frac{\frac{1}{3} t}{\frac{1}{3} h} = \frac{t}{h}.$$

But we found above that

$$\frac{P}{W} = \frac{\gamma_a h}{\gamma_m t}.$$

Equating the two expressions for $\frac{P}{W}$ gives

$$\frac{t}{h} = \frac{\gamma_a h}{\gamma_m t},$$

or

$$\left(\frac{t}{h}\right)^2 = \frac{\gamma_a}{\gamma_m},$$

and consequently

$$\frac{t}{h} = \sqrt{\frac{\gamma_a}{\gamma_m}}.$$

Therefore $\frac{t}{h}$, which is equal to the slope of the down-stream face of the dam, must be made equal to the square root of the specific weights of water and masonry. Consequently the thickness $t$ of the base of the triangular dam will be given by

$$t = \sqrt{\frac{\gamma_a}{\gamma_m}}\, h.$$

**54. Trapezoidal Profile.** — In actual practice it is impossible to use the triangular dam profile, since it is necessary to give the dam a certain thickness at the top. Therefore the simplest profile is the trapezoidal one, as exhibited by Fig. 68. Here *ABCD* is the cross-section of the dam, $t'$ being the thickness of the structure at the top. Using the same notation as in the preceding case, the pressure of

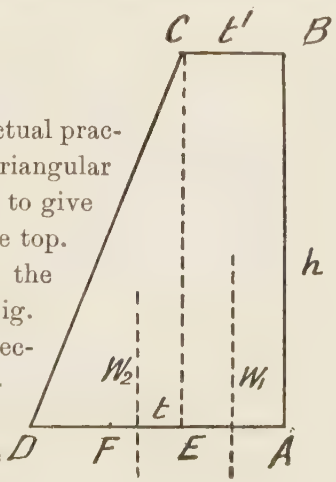



Fig. 68.

the water is

$$P = \tfrac{1}{2}\,\gamma_a h^2,$$

and the weight of the dam is

$$W m = \tfrac{1}{2}\,(t + t')\,h\gamma_m.$$

The weight $W$ can be considered as the sum of the weights of the rectangular prism $ABCE$ and the triangular prism $DEC$. Call the weights of these two prisms $W_1$ and $W_2$. Then we will have

$$W = W_1 + W_2,$$

$$W_1 = \gamma_m t' h,$$

and $$W_2 = \tfrac{1}{2}\,\gamma_m\,(t + t')\,h.$$

Let $F$ be the outer edge of the middle third of the base $AD$. The distance of $W_1$ from $F$ will be

$$w_1 = FW_1 = \frac{2}{3}\,t - \frac{1}{2}\,t' = \frac{4\,t - 3\,t'}{6},$$

while the distance of $W_2$ from $F$ will be

$$w_2 = FW_2 = \frac{2}{3}\,t - t' - \frac{t - t'}{3} = \frac{2\,t - 3\,t' - t + t'}{3},$$

$$w_2 = \frac{t - 2\,t'}{3}.$$

For equilibrium about the point $F$ we must have the algebraic sum of the moments equal to zero, or

$$P = \tfrac{1}{3}\,h - W_1 w_1 - W_2 w_2 = 0.$$

Substituting in this equation the various values found above, we obtain

$$\frac{1}{3} \times \frac{1}{2}\,\gamma_a h^2 \times h - \left(\gamma_m t' h \times \frac{4\,t - 3\,t'}{6}\right) - \left(\frac{1}{2}\,\gamma_m\,(t - t')\,h \times \frac{t - 2\,t'}{3}\right) = 0.$$

Simplifying,

$$\tfrac{1}{6}\,\gamma_a h^3 - \tfrac{1}{6}\,\gamma_m t' h(4\,t - 3\,t') - \tfrac{1}{6}\,\gamma_m (t - t')h(t - 2\,t') = 0.$$

Now write $\alpha t$ in place of $t'$, so as to represent $t'$ in terms of $t$, $\alpha$ being a coefficient. Then

$$\tfrac{1}{6}\gamma_a h^3 - \tfrac{1}{6}\alpha t h(4t - 3\alpha t) - \tfrac{1}{6}\gamma_m(t - \alpha t)h(t - 2\alpha t) = 0.$$

This reduces as follows:

$$\gamma_a h^2 - \gamma_m \alpha t(4t - 3\alpha t) - \gamma_m(t - \alpha t)(t - 2\alpha t) = 0,$$

$$\gamma_a h^2 - \gamma_m \alpha t^2 4 + \gamma_m \alpha^2 t^2 \times 3 - \gamma_m(t^2 - \alpha t^2 - 2\alpha t^2 + 2\alpha^2 t^2) = 0,$$

$$\gamma_a h^2 - \gamma_m t^2(4\alpha - 3\alpha^2 + 1 - \alpha - 2\alpha + 2\alpha^2) = 0,$$

$$\gamma_a h^2 - \gamma_m t^2(\alpha + 1 - \alpha^2) = 0,$$

or

$$t = \frac{\sqrt{\dfrac{\gamma_a}{\gamma_m}}}{\sqrt{\alpha + 1 - \alpha^2}} h.$$

which is equal to the thickness of base required for a profile, as found above, divided by the quantity $\sqrt{1 + \alpha - \alpha^2}$. For both the values $\alpha = 0$ and $\alpha = 1$ the quantity $\sqrt{1 + \alpha - \alpha^2}$ is equal to 1. This means that a triangular and a rectangular profile require the same base width. In passing from the one to the other, the quantity referred to increases to a certain limit from which it decreases again. Its greatest value corresponds to $\alpha = 0.5$. This means that the thickness $t$ of the base of the dam decreases with increase of $\alpha$ from 0 to 0.5, while it increases when the value of $\alpha$ lies between 0.5 and 1. In other words, it will be most economical to give to the top of the dam a thickness less than half the width of the base; greater thickness of top produces a waste of masonry.

**55. Pentagonal Profile.** — In the trapezoidal profile there is an excess of stability at the section near the top of the dam, which means that a large quantity of masonry is used that might be saved by a better shape of cross-section.

Since the best theoretical profile is the triangular one, which is objectionable only because it gives zero thickness at the top, we may retain approximate theoretical perfection and yet correct this defect by using a cross-section whose lower part is similar to the corresponding part of the triangular section while the upper part is a wall of constant thickness. The result of this modification is the pentagonal profile, Fig. 69.

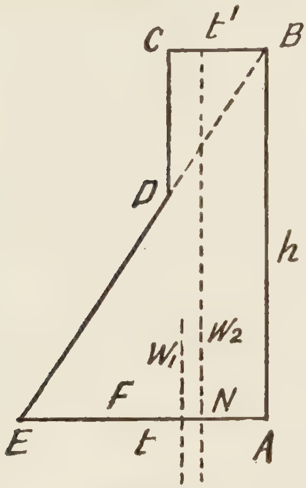


Fig. 69.

*ABCDE*, the profile of the dam, can be considered as composed of the triangle *ABE* of altitude $h$ and base $t$, and the triangle *BCD* of altitude $h'$ and base $t'$. The pressure of water will be as before,

$$P = \tfrac{1}{2}\, \gamma_a h^2,$$

the weight of the dam will be equal to the weight of the two triangular prisms, $W = W_1 + W_2$;

since $$W_1 = \tfrac{1}{2}\, \gamma_m th,$$

and $$W_2 = \tfrac{1}{2}\, \gamma_m t'h',$$

we will have

$$W = \tfrac{1}{2}\, \gamma_m th + \tfrac{1}{2}\, \gamma_m t'h'.$$

The right triangles *AEB* and *BCD* are similar, having their angles equal and their sides parallel; consequently their sides will be proportional,

$$\frac{h}{h'} = \frac{t}{t'},$$

or

$$h' = \frac{t'}{t} h.$$

Expressing the dimensions of the upper section $BCD$ in terms of the width of base by making $t' = \alpha t$, we obtain,

$$W = \frac{1}{2} \gamma_m th + \frac{1}{2} \gamma_m \alpha t \times \frac{\alpha t}{t} h,$$

or

$$W = \tfrac{1}{2} \gamma_m th + \tfrac{1}{2} \gamma_m \alpha^2 th.$$

For stability against overturning it is necessary that the algebraic sum of the moments about the outer edge of the middle third of the base be zero. That is, using the distances $w_1$ and $w_2$ in the same sense as before,

$$P \times \tfrac{1}{3} h - W_1 w_1 - W_2 w_2 = 0.$$

Now

$$w_1 = \tfrac{1}{3} t \text{ and } w_2 = FA - NA = \tfrac{2}{3} t - \tfrac{2}{3} \alpha t = \tfrac{2}{3} t(1 - \alpha),$$

so that the equation of moments becomes,

$$\tfrac{1}{2} \gamma_a h^2 \times \tfrac{1}{3} h - \tfrac{1}{2} \gamma_m th \times \tfrac{1}{3} t - \tfrac{1}{2} \gamma_m \alpha^2 th \times \tfrac{2}{3} t(1 - \alpha) = 0,$$

or, reducing,

$$\tfrac{1}{6} \gamma_a h^3 - \tfrac{1}{6} \gamma_m t^2 h - \tfrac{2}{6} \gamma_m \alpha^2 (1 - \alpha) t^2 h = 0,$$

or

$$\gamma_a h^2 - \gamma_m t^2 (1 + 2 \alpha^2 (1 - \alpha)) = 0,$$

from which

$$t^2 = \frac{\gamma_a h^2}{\gamma_m (1 + 2 \alpha^2 (1 - \alpha))},$$

or

$$t = \frac{\sqrt{\dfrac{\gamma_a}{\gamma_m}}}{\sqrt{1 + 2 \alpha^2 - 2 \alpha^3}} h,$$

which is the required thickness of base for the pentagonal profile.

Dividing both terms by $h$,

$$\frac{t}{h} = \frac{\sqrt{\frac{\gamma_a}{\gamma_m}}}{\sqrt{1 + 2\,\alpha^2 - 2\,\alpha^3}},$$

which is the slope $EB$. It is equal to the slope of the pure triangular profile, $\sqrt{\frac{\gamma_a}{\gamma_m}}$, divided by the quantity $\sqrt{1 + 2\,\alpha^2 - 2\,\alpha^3}$. This quantity is equal to 1 when $\alpha = 0$ and when $\alpha = 1$, *i.e.* for the pure triangular and the pure rectangular profile. By differentiation, $\sqrt{1 + 2\,\alpha^2 - 2\,\alpha^3}$ with respect to $\alpha$, we may find what value of $\alpha$ gives the greatest value to the denominator, and consequently the smallest width of base. It is found to be 0.666, that is, the smallest bottom width $t$ is obtained when $t'$, the top width, is made two thirds of the bottom width.

**56.** In the following tables the various dimensions and values for the minimum trapezoidal and pentagonal profiles are given in terms of the height. In working out these tables the ratio of upper base to lower base was taken at successive tenths from 0 to 1. The weights of the materials were assumed to be

water, $\gamma_a = 62.5$ lbs. per cubic foot.

masonry, $\gamma_m = 150$ lbs. per cubic foot.

| Number | Ratio, $\frac{t'}{t} = a$ | Thickness of Base, $t$ | Thickness of Top, $t'$ | Volume, $V = \frac{t+t'}{2} h$ | Water Pressure, $P = \frac{1}{2} Y_a h^2$ | Weight, $W = \frac{t+t'}{2} Y_m h$ | Coefficient of Friction, $\frac{P}{W}$ | Remarks |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.00 | 0.6454 $h$ | 0.00 $h$ | .3227 $h^2$ | 31.25 $h^2$ | 48.405 $h^2$ | .645 | Triangular profile |
| 2 | 0.1 | 0.592 $h$ | 0.0592 $h$ | .3256 $h^2$ | 31.25 $h^2$ | 48.84 $h^2$ | .639 | Trapezoidal profile |
| 3 | 0.2 | 0.547 $h$ | 0.1094 $h$ | .3282 $h^2$ | 31.25 $h^2$ | 49.23 $h^2$ | .634 | " " |
| 4 | 0.3 | 0.533 $h$ | 0.1599 $h$ | .3464 $h^2$ | 31.25 $h^2$ | 51.967 $h^2$ | .601 | " " |
| 5 | 0.4 | 0.524 $h$ | 0.2096 $h$ | .3668 $h^2$ | 31.25 $h^2$ | 55.02 $h^2$ | .566 | " " |
| 6 | 0.5 | 0.516 $h$ | 0.258 $h$ | .387 $h^2$ | 31.25 $h^2$ | 58.05 $h^2$ | .538 | " " |
| 7 | 0.6 | 0.524 $h$ | 0.3144 $h$ | .4192 $h^2$ | 31.25 $h^2$ | 62.88 $h^2$ | .496 | " " |
| 8 | 0.7 | 0.533 $h$ | 0.3731 $h$ | .4530 $h^2$ | 31.25 $h^2$ | 67.9575 $h^2$ | .459 | " " |
| 9 | 0.8 | 0.547 $h$ | 0.4376 $h$ | .4923 $h^2$ | 31.25 $h^2$ | 73.845 $h^2$ | .423 | " " |
| 10 | 0.9 | 0.592 $h$ | 0.5328 $h$ | .5024 $h^2$ | 31.25 $h^2$ | 84.36 $h^2$ | .370 | " " |
| 11 | 1.00 | 0.6454 $h$ | 0.6454 $h$ | .6454 $h^2$ | 31.25 $h^2$ | 96.81 $h^2$ | .322 | Rectangular profile |

| Number | Ratio, $\frac{t'}{t} = a$ | Thickness of Base, $t$ | Thickness of Top, $t'$ | Volume, $V = \frac{1}{2}(1 + a^2)th$ | Water Pressure, $P = \frac{1}{2} Y_a h^2$ | Weight, $W = V \cdot Y_m$ | Coefficient of Friction, $\frac{P}{W}$ | Remarks |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.0 | .6454 $h$ | .0 | .3227 $h^2$ | 31.25 $h^2$ | 48.40 $h^2$ | .645 | Triangular profile |
| 2 | 0.1 | .633 $h$ | .063 $h$ | .3196 $h^2$ | 31.25 $h^2$ | 47.94 $h^2$ | .651 | Pentagonal profile |
| 3 | 0.2 | .625 $h$ | .124 $h$ | .3250 $h^2$ | 31.25 $h^2$ | 48.75 $h^2$ | .641 | " " |
| 4 | 0.3 | .607 $h$ | .180 $h$ | .3308 $h^2$ | 31.25 $h^2$ | 49.62 $h^2$ | .629 | " " |
| 5 | 0.4 | .580 $h$ | .232 $h$ | .3364 $h^2$ | 31.25 $h^2$ | 50.46 $h^2$ | .619 | " " |
| 6 | 0.5 | .573 $h$ | .285 $h$ | .3581 $h^2$ | 31.25 $h^2$ | 53.71 $h^2$ | .581 | " " |
| 7 | 0.6 | .564 $h$ | .336 $h$ | .3835 $h^2$ | 31.25 $h^2$ | 57.52 $h^2$ | .543 | " " |
| 8 | 0.66 | .562 $h$ | .369 $h$ | .4044 $h^2$ | 31.25 $h^2$ | 60.66 $h^2$ | .515 | " " |
| 9 | 0.7 | .563 $h$ | .392 $h$ | .4144 $h^2$ | 31.25 $h^2$ | 62.16 $h^2$ | .502 | " " |
| 10 | 0.8 | .572 $h$ | .456 $h$ | .4685 $h^2$ | 31.25 $h^2$ | 70.27 $h^2$ | .444 | " " |
| 11 | 0.9 | .597 $h$ | .531 $h$ | .5402 $h^2$ | 31.25 $h^2$ | 81.03 $h^2$ | .385 | " " |
| 12 | 1.0 | .6454 $h$ | .645 $h$ | .6454 $h^2$ | 31.25 $h^2$ | 96.81 $h^2$ | .322 | Rectangular profile |

## Practical Cross-sections

**57.** The theoretical dam profiles examined in the preceding chapter are very seldom used for practical purposes. But the trapezoidal and pentagonal profiles are valuable guides to the engineer in designing suitable cross-sections for dams. The trapezoidal profile approximates the preferred outline for submerged dams, while the pentagonal profile is the basal form of all the high dams erected in recent years.

**58. Submerged dams. Ogee Profile.** — According to Professor Merriman, the pressure of water against a submerged dam is given by the formula

$$P = \tfrac{1}{2}\, h\, (h + d),$$

where $d$ is the height of water above the dam. The point of application of the resultant pressure on the upstream side of the dam is at a height above the base given by the formula,

$$p = \frac{h + 3d}{h + 2d} \times \frac{1}{3}\ d.$$

The most suitable profile for a submerged dam is the one having an ogee curve on the downstream side which guides the water flowing over the crest. Such a profile is produced by changing the downstream face to the form of a reverse curve tangent to the face at mid-height, the upper curve giving a rounded top to the dam while the lower one turns forward horizontally downstream.

In designing a submerged dam where the depth of water over the crest will not be greater than 10 ft., the following practical rule may be used with advantage. It gives a profile in which the line of resistance is clearly shown in Fig. 71, and it is very close to the median line of the structure.

Determine, first, the thickness of base for a trapezoidal

profile according to the method given previously, taking as depth of water the height of dam plus the depth of water over the crest, or $H = h + d$, and making the top width 0.3 of the width of the base; in other words, making $a = 0.3$. The trapezoidal profile $ABDC$ (Fig. 70) will thus be obtained.

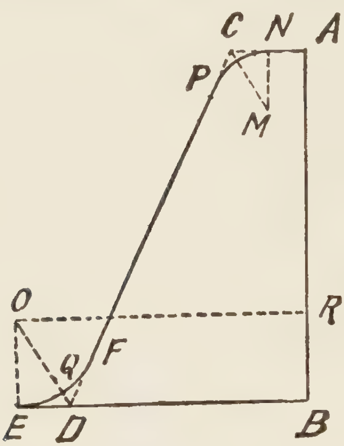



FIG. 70.

Bisect the angle $ACD$ by the line $CM$. Then from the middle point of $AC$ drop the vertical $N$ meeting the bisector at $M$. With $M$ as center and $MN$ as radius describe the arc $NP$. Similarly bisect the angle $EDF$ by the line $DO$. From a point $R$ on the back of the dam, at a height equal to $4\,d$ above the base, draw a horizontal line $RO$ meeting the bisector at $O$. With $O$ as center and $OE$ as radius describe the arc $EQF$. The result is the profile $ANPFQEB$, which is a suitable cross-section for submerged dams when the height of water above the dam or $d$ is not more than 10 ft.

The resistance of an ogee profile designed according to the preceding rule was determined graphically by Mr. William J. S. Deevy, and the result of his work is given in the following table, referring to Fig. 71. The data assumed were as follows:

$h$ = height of dam = 80 ft.

$d$ = depth of water over the crest = 5 ft.

$\gamma_a$ = weight of water per cubic foot = $62\frac{1}{2}$ lbs.

$\gamma_m$ = weight of masonry per cubic foot = 150 lbs.

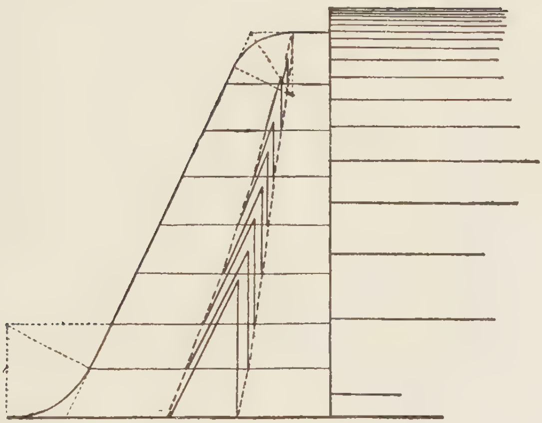


Fig. 71.

| Number of Section from Crest | Length of Joint | Water Pressure on Face above Section, Lbs. | Area of Cross-section | Weight of Dam above Section, Lbs. | Height of Center of Pressure, Ft. | Distance of Center Line either Face | Distance of Resultant from Downstream Face | Distance of Center of Gravity from Upstream Face |
|---|---|---|---|---|---|---|---|---|
| 1 | 21.5 | 6250 | 173.33 | 26000 | 4.16 | 10.5 | 11.5 | 9 |
| 2 | 26 | 18750 | 410.83 | 61625 | 11 | 13 | 13 | 10.5 |
| 3 | 31 | 37500 | 695.83 | 104375 | 11.25 | 15.5 | 15 | 12 |
| 4 | 36 | 62500 | 1030.83 | 154625 | 14.66 | 18 | 16.7 | 13.5 |
| 5 | 41 | 93750 | 1415.83 | 212375 | 18.05 | 20.5 | 17.5 | 14.5 |
| 6 | 46 | 131250 | 1850.83 | 277625 | 21.43 | 23 | 19.5 | 16 |
| 7 | 50.5 | 175000 | 2333.33 | 350000 | 24.8 | 25.75 | 20.5 | 17.5 |
| 8 base | 67.5 | 224000 | 2923.33 | 438500 | 28.1 | 33.75 | 35.5 | 19.5 |

**59. High Dams.** — Many authors have suggested practical rules for the designing of high dams. All these profiles, however, closely resemble the pentagonal profile, with the difference that the length of the horizontal joints is increased either by giving a batter to the upstream side of the dam or by making the face curvilinear, while on the downstream

face the vertical and inclined portions are connected by a curve of small radius. In such profiles the line of resistance, both with full and with empty reservoir, falls within the middle third of the joint at all joints.

**60. Crugnola's Section.** — Mr. Giacomo Crugnola, in his work, "Sui muri di sostegno delle terre e sulle traverse dei serbatoi d'acqua," gives the following data for designing masonry dams, resulting in the profile shown by Fig. 72. In this table the various dimensions by which the profile is determined are given in meters, and the volumes of masonry are given in cubic meters.

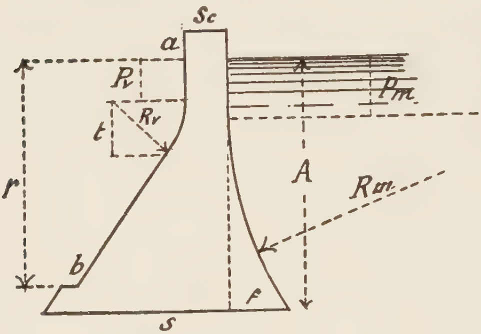


FIG. 72.

| | | | Upstream Side | | | Downstream Side | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *A* | *a* | *Sc* | *Pm* | *Rm* | *F* | *Pv* | *Rv* | *t* | *r* | *s* | *b* | Volume |
| 5 | 0.5 | 1.7 | 3.5 | 16 | 0.07 | 1.7 | 4 | 2.75 | 5 | 2.52 | | 10.500 |
| 10 | 0.9 | 2 | 6.0 | 24 | 0.34 | 1.8 | 6 | 4.75 | 10 | 6.08 | | 35.055 |
| 15 | 1.3 | 2.3 | 7.0 | 32 | 1.02 | 1.9 | 8 | 6.00 | 15 | 9.81 | | 76.637 |
| 20 | 1.5 | 2.5 | 8.0 | 40 | 1.84 | 2.0 | 10 | 7.25 | 20 | 13.70 | | 133.010 |
| 25 | 2 | 3.00 | 9.0 | 48 | 2.75 | 2.1 | 12 | 8.50 | 25 | 17.99 | | 217.700 |
| 30 | 2.4 | 3.5 | 10.0 | 56 | 3.69 | 2.2 | 14 | 9.50 | 30 | 21.75 | | 314.557 |
| 35 | 2.8 | 4.0 | 11.0 | 64 | 4.47 | 2.3 | 16 | 11.50 | 35 | 27.90 | | 455.804 |
| 40 | 3.00 | 4.25 | 12.0 | 72 | 5.67 | 2.4 | 18 | 13.0 | 35 | 34.04 | 1 | 610.442 |
| 45 | 3.25 | 4.50 | 13.5 | 80 | 6.46 | 2.5 | 20 | 14.5 | 35 | 38.88 | 1 | 781.423 |
| 50 | 3.50 | 4.75 | 15.0 | 88 | 7.26 | 2.6 | 22 | 16.0 | 35 | 46.92 | 1 | 996.108 |

**61. Krantz's Section.** — The French engineer Krantz, in his book, "Étude sur les murs de réservoirs," has suggested another practical profile for designing dams of various heights. Krantz's profile is shown in Fig. 73. All the elements for designing the Krantz profile for any height are given in the following table in meters:

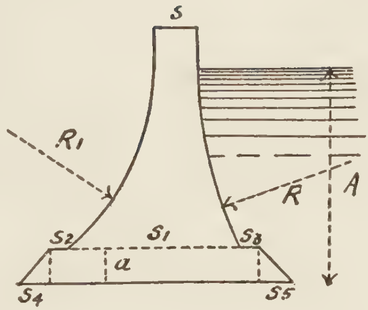



Fig. 73.

| $A$ | $a$ | $s$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | $R$ | $R_1$ | Volume |
|---|---|---|---|---|---|---|---|---|---|---|
| 5 | | 2.0 | 4 | | | | | 13 | 13 | 14.280 |
| 10 | | 2.5 | 7 | | | | | 26 | 21.35 | 42.190 |
| 15 | | 3.0 | 10.5 | | | | | 39 | 27.25 | 85.960 |
| 20 | | 3.5 | 14.5 | | | | | 52 | 32.07 | 147.660 |
| 25 | | 4.0 | 19 | | | | | 65 | 36.25 | 229.120 |
| 30 | | 4.5 | 24 | | | | | 78 | 40.08 | 331.890 |
| 35 | | 5.0 | 29.5 | | | | | 91 | 43.75 | 457.290 |
| 40 | 5 | 5.0 | 29.5 | 1 | 1 | 5 | 3.33 | 91 | 43.75 | 635.620 |
| 45 | 10 | 5.0 | 29.5 | 1 | 1 | 10 | 6.67 | 91 | 43.75 | 855.640 |
| 50 | 15 | 5.0 | 29.5 | 1 | 1 | 15 | 10 | 91 | 43.75 | 1117.290 |

Both Crugnola and Krantz determined their profiles by a long series of trials designed without the use of any formula or definite rule.

**62. Author's Section.** — A practical profile for a dam of any height may be easily obtained in the following manner, based upon the theoretical profile. The method can be applied to any case.

Calculate the thickness of base for a pentagonal profile (Fig. 69) by the methods previously given, assuming for $a$ the value of 0.2, so that $t' = 0.2\,t$. From the reëntrant angle

of the downstream face, where the vertical and inclined portions of the face meet, draw a horizontal line, and with center on this line, upstream from the dam, and a radius equal to three times the height of the dam, describe an area of circle tangent to the upstream face; this curve will form the back of the dam in place of the vertical face of the theoretical profile. Then join the vertical and inclined portions of the downstream face by a curve of radius equal to the top width of the dam. There results a profile similar to the one shown in Fig. 74.

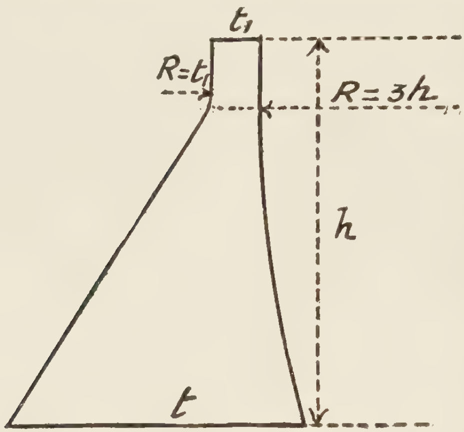


Fig. 74.

The lines of resistance of a dam 100 ft. high designed according to the method indicated above are given in Fig. 75, both for reservoir full and for reservoir empty. In the calculation of these lines, the weight of water was assumed to be 625 lbs., and the weight of masonry 150 lbs. per cubic foot. The dimensions and pressures for the various joints, as given in the following table, were calculated by Mr. John M. Fitzgerald:

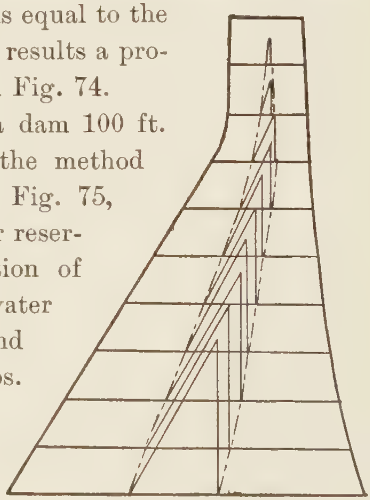


Fig. 75.

| Section | Water Pressure, Lbs. | Weight of Dam above, Lbs. | Length of Joint, Ft. | Distance from Front Face to Line of Resultant Pressure, Ft. | Distance from Resultant to Center of Gravity, Ft. | Distance from Center of Gravity to Upstream Face, Ft. |
|---|---|---|---|---|---|---|
| 1 | 3115 | 22500 | 13.5 | 6.0 | 0.5 | 7.0 |
| 2 | 12500 | 45250 | 16.5 | 7.0 | 2.5 | 7.5 |
| 3 | 28125 | 72500 | 19.5 | 7.0 | 4.5 | 8.0 |
| 4 | 50000 | 107500 | 27 | 11.5 | 6.0 | 9.5 |
| 5 | 78125 | 155000 | 34 | 14.5 | 7.5 | 12 |
| 6 | 112500 | 207.500 | 42 | 16.5 | 10.5 | 15.0 |
| 7 | 153125 | 280.000 | 49.5 | 19.0 | 12.5 | 18.0 |
| 8 | 200000 | 350000 | 56.5 | 21.0 | 15.5 | 20.0 |
| 9 | 253.125 | 420000 | 64.5 | 23.5 | 17.0 | 24.0 |
| 10 | 312.500 | 556250 | 73.5 | 25.5 | 18.0 | 34.0 |

Comparing the figures in the last three columns, it will be seen that the lines of resistance, both reservoir full and reservoir empty, falls well within the middle third of the corresponding joint.

In very high dams the value of $\alpha$ for this profile ($\alpha = t' \div t$, or top width divided by width of base) may even be decreased to 0.15 or 0.1; if this is not done, the dam will be very thick at the top, with corresponding waste of masonry.

Dams built for reservoir purposes must be higher than the level of water in the pool formed by the dam. The portion of the structure which rises above the water is called the crest of the dam. It is noteworthy that both Krantz and Crugnola, as well as all others who have suggested dam profiles, have given various dimensions for the height of crest of dams of various heights. As a general rule, it is well to make the crest at least $\frac{1}{10}$ the total height of the dam. In the profile suggested by the author, the crest was not taken into account, but following the general

rule it would be very easy to add to the profile of Fig. 75 a prism of masonry 10 ft. high. Since no water pressure is added hereby, the added prism of masonry tends to increase the stability of the structure, since it increases the weight upstream of the center of gravity.

In regard to the manner of determining graphically the line of resistance in a dam of a given profile, the following method may be used: divide the profile into a number of sections by drawing horizontal lines 5 ft. or 10 ft. apart. These lines will divide the profile into so many dams with parallel bases, in each case considering the entire area above the horizontal line or joint in question. Find the center of gravity of each of these small dams and draw vertical lines through the various centers of gravity. Mark the point where each vertical cuts its corresponding base, and draw a continuous line through the points thus obtained. This is the line of resistance for empty reservoir, or when the dam has no pressure to resist. Next draw a force polygon, laying off vertically in succession the weight of the successive slices of the dam. From the upper end of the first-weight line in the force polygon draw a horizontal line and upon it lay off, to the same scale as used for the weights, the successive water pressures on the various slices. Connect the ends of the pressure and weight lines corresponding to each other. The connecting lines represent to scale the resultants of weight and pressure for the several sections of the dam, in each case the entire section above the joint considered. Then return to the profile, and at $\frac{1}{3}$ of the height of each section or partial dam (point of application of the pressure) draw from the vertical which was dropped from the center of gravity a line parallel to the corresponding resultant. Mark the points where the various resultants intersect the

corresponding points, and connect these points by a continuous line. This is the line of resistance for full reservoir, *i.e.* when the dam is resisting the maximum water pressure. The dash-and-dot lines in Fig. 75 are the two lines of resistance, the right-hand one being for reservoir empty and the left-hand one for reservoir full. The gravity lines and the lines of resultant pressure for each dam of partial height are also shown in the figure, showing clearly how the lines of resistance are obtained.